ECO227Y5 Tutorial 10

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Question 5.27

Question: In Exercise 5.9, we determined that

$$f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

is a valid joint probability density function. Find

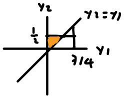
(a) the marginal density functions for Y_1 and Y_2.

(b) $P\left(Y_2 \le \frac{1}{2} \mid Y_1 \le \frac{3}{4}\right)$.

$0 \le y_1 \le y_2 \le 1$ integrate y_2's bounded

a.) $f_{y_1} = \int_{-\infty}^{\infty} 6(1-y_2)\,dy_2 = \int_{y_1}^{1} 6 - 6y_2\,dy_2 = 6y_2 - 3y_2^2 \Big|_{y_1}^{1} = 3(1-y_1)^2 \Rightarrow f_{y_1} = \begin{cases} 3(1-y_1)^2, & 0 \le y_1 \le 1 \\ 0, & \text{otherwise} \end{cases}$

$f_{y_2} = \int_{0}^{y_2} 6(1-y_2)\,dy_1 = 6y_2(1-y_2) \Rightarrow f_{y_2} = \begin{cases} 6y_2(1-y_2), & 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$



b.) $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)} \Rightarrow P\left(Y_2 \le \frac{1}{2} \mid Y_1 \le \frac{3}{4}\right) = \dfrac{\int_0^{\frac{1}{2}} \int_{y_1}^{\frac{1}{2}} \overbrace{6(1-y_2)}^{f(y_1,y_2)}\,dy_2\,dy_1}{\int_0^{\frac{3}{4}} \underbrace{3(1-y_1)^2}_{f_1(y_1)}\,dy_1}$ ← u-sub

$\Rightarrow P\left(Y_2 \le \frac{1}{2} \mid Y_1 \le \frac{3}{4}\right) = \dfrac{\frac{1}{2}}{\frac{63}{64}} = \dfrac{32}{63}$

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is a valid joint probability density function. Find

(c) the conditional density function of Y_1 given $Y_2 = y_2$.

(d) the conditional density function of Y_2 given $Y_1 = y_1$.

(e) $P\left(Y_2 \ge \frac{3}{4} \mid Y_1 = \frac{1}{2}\right)$.

c.) $f(y_1 \mid y_2) = \dfrac{f(y_1, y_2)}{f_2(y_2)} = \dfrac{6(1-y_2)}{6y_2(1-y_2)} = \dfrac{1}{y_2} \Rightarrow f(y_1 \mid y_2) = \begin{cases} \frac{1}{y_2}, & 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$

d.) $f(y_2 \mid y_1) = \dfrac{f(y_1, y_2)}{f_1(y_1)} = \dfrac{6(1-y_2)}{3(1-y_1)^2} = \dfrac{2(1-y_2)}{(1-y_1)^2} \Rightarrow f(y_2 \mid y_1) = \begin{cases} \frac{2(1-y_2)}{(1-y_2)^2}, & y_1 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$

e.) $P(Y_2 \ge \frac{3}{4} \mid Y_1 = \frac{1}{2}) = \displaystyle\int_{\frac{3}{4}}^{1} \frac{2(1-y_2)}{(1-\frac{1}{2})^2}\, dy_2 = \int_{\frac{3}{4}}^{1} \frac{2(1-y_2)}{\frac{1}{4}}\, dy_2 = \int_{\frac{3}{4}}^{1} 8 - 8y_2\, dy_2 = 8y_2 - 4y_2^2 \Big|_{\frac{3}{4}}^{1} = \frac{1}{4}$

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Question 5.32

Question: Suppose that the random variables Y_1 and Y_2 have joint probability density function $f(y_1, y_2)$ given by (see Exercise 5.14)

$$f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \le y_1 \le y_2,\ y_1 + y_2 \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that the marginal density of Y_1 is a beta density with $\alpha = 3$ and $\beta = 2$.

(b) Derive the marginal density of Y_2.

$y_1 + y_2 \le 2 \Rightarrow y_2 \le 2 - y_1$, $y_2 \ge y_1$

combine: $y_1 \le 2 - y_1 \Rightarrow y_1 \le 1$

a.) $f_{Y_1} = \int_{y_1}^{2-y_1} 6y_1^2 y_2 \, dy_2 = 3y_1^2 y_2^2 \Big|_{y_1}^{2-y_1} = 3y_1^2[(2-y_1)^2 - y_1^2] = 3y_1^2(4(1-y_1)) \Rightarrow f_1(y_1) = \begin{cases} 12y_1^2(1-y_1), & 0 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$

$f_{Y_1} = \frac{1}{B(3,2)} y^{3-1}(1-y)^{2-1} \Rightarrow B(3,2)$

There is a split in y_2

b.) For $0 \le y_2 \le 1$: $0 \le y_1 \le y_2$
$1 \le y_2 \le 2$: $0 \le y_1 \le 2 - y_2$ $\Rightarrow f_{Y_2} = \begin{cases} \int_0^{y_2} 6y_1^2 y_2 \, dy_1, & 0 \le y_2 \le 1 \\ \int_0^{2-y_2} 6y_1^2 y_2 \, dy_1, & 1 \le y_2 \le 2 \\ 0, & \text{elsewhere} \end{cases} \Rightarrow f_{Y_2} = \begin{cases} 2y_2^4, & 0 \le y_2 \le 1 \\ 2y_2(2-y_2)^3, & 1 \le y_2 \le 2 \\ 0, & \text{elsewhere} \end{cases}$

Question 5.32

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$$f(y_1, y_2) = \begin{cases} 6y_1^2 y_2, & 0 \le y_1 \le y_2, \; y_1 + y_2 \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(c) Derive the conditional density of Y_2 given $Y_1 = y_1$.

(d) Find $P(Y_2 < 1.1 \mid Y_1 = 0.60)$.

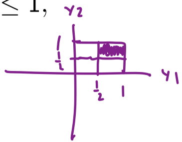
c.) $$f(y_2 \mid y_1) = \frac{f(y_1, y_2)}{f_{y_1}(y_1)} = \frac{6y_1^2 y_2}{12 y_1^2 (1-y_1)} = \frac{y_2}{2(1-y_1)} \Rightarrow \begin{cases} \frac{y_2}{2(1-y_1)}, & 0 \le y_1 \le y_2 \le 2 - y_1 \\ 0, & \text{elsewhere} \end{cases}$$

d.) $$P(Y_2 < 1.1 \mid Y_1 = 0.60) = \int_{0.60}^{1.1} \frac{y_2}{2(0.4)} \, dy_2 = \frac{5}{8} y_2^2 \Big|_{0.60}^{1.1} = \frac{17}{32}$$

Question 5.36

Question: In Exercise 5.16, Y_1 and Y_2 denoted the proportions of time during which employees I and II actually performed their assigned tasks during a workday. The joint density of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$



(a) Find the marginal density functions for Y_1 and Y_2.

(b) Find $P\left(Y_1 \ge \frac{1}{2} \mid Y_2 \ge \frac{1}{2}\right)$.

a.) $f_{Y_1} = \int_0^1 y_1 + y_2 \, dy_2 = y_1 y_2 + \frac{y_2^2}{2} \Big|_0^1 = y_1 + \frac{1}{2} \Rightarrow f_{Y_1} = \begin{cases} y_1 + \frac{1}{2}, & 0 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$

$f_{Y_2} = \int_0^1 y_1 + y_2 \, dy_1 = \frac{y_1^2}{2} + y_2 y_1 \Big|_0^1 = \frac{1}{2} + y_2 \Rightarrow f_{Y_2} = \begin{cases} y_2 + \frac{1}{2}, & 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$

b.) $p(Y_1 \ge \frac{1}{2}, Y_2 \ge \frac{1}{2}) = \dfrac{P(Y_1 \ge \frac{1}{2}, Y_2 \ge \frac{1}{2})}{P(Y_2 \ge \frac{1}{2})} = \dfrac{\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 y_1 + y_2 \, dy_1 \, dy_2}{\int_{\frac{1}{2}}^1 y_2 + \frac{1}{2} \, dy_2} = \dfrac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}$

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Question: In Exercise 5.16, Y_1 and Y_2 denoted the proportions of time during which employees I and II actually performed their assigned tasks during a workday. The joint density of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} y_1 + y_2, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(c) If employee II spends exactly 50% of the day working on assigned duties, find the probability that employee I spends more than 75% of the day working on similar duties.

$P(Y_1 \ge \frac{3}{4} \mid Y_2 = \frac{1}{2})$

Find $f(y_1 \mid y_2) = \dfrac{f(y_1, y_2)}{f_{Y_2}(y_2)} = \begin{cases} \dfrac{y_1 + y_2}{\frac{1}{2} + y_2}, & 0 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases}$

$$P(Y_1 \ge \tfrac{3}{4} \mid Y_2 = \tfrac{1}{2}) = \int_{\frac{3}{4}}^{1} y_1 + \tfrac{1}{2} \, dy_1 = \frac{y_1^2}{2} + \frac{y_1}{2} \Big|_{\frac{3}{4}}^{1} = \frac{11}{32}$$

Question 5.49

Question: In Example 5.4 and Exercise 5.5, we considered the joint density of Y_1, the proportion of the tank's capacity stocked at the beginning of the week, and Y_2, the proportion sold during the week:

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that Y_1 and Y_2 are *dependent*.

$f_{Y_1} = \int_0^{y_1} 3y_1 \, dy_2 = 3y_1^2$

Recall: Y_1, Y_2 independent if $f(y_1, y_2) = f_1(y_1) \cdot f_2(y_2)$ for all values of (y_1, y_2)

$f_{Y_2} = \int_{y_2}^{1} 3y_1 \, dy_1 = \frac{3}{2} y_1^2 \Big|_{y_2}^{1} = \frac{3}{2}(1 - y_2^2)$

clearly $f(y_1, y_2) \neq f_1(y_1) \cdot f_2(y_2)$ at all supported (y_1, y_2) values

Ex. $y_1 = 1$, $y_2 = 0$

$f_1(1) = 3$, $f_2(0) = \frac{3}{2} \Rightarrow f_1 \cdot f_2 = \frac{9}{2}$

$f(1, 0) = 3 \neq \frac{9}{2} \Rightarrow$ Dependent.

Question 5.64

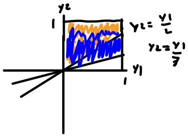
Question: Let Y_1 and Y_2 be independent random variables that are both uniformly distributed on the interval $(0, 1)$. Find

$$P(Y_1 < 2Y_2 \mid Y_1 < 3Y_2).$$

$Y_1, Y_2 \sim \text{Uni}(0,1)$ and independent so:

$$f_1(y_1) = \begin{cases} 1 & 0 \le y_1 \le 1 \\ 0, & \text{elsewhere} \end{cases} \qquad f_2(y_2) = \begin{cases} 1 & 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f(y_1, y_2) = f_1(y_1) \cdot f_2(y_2) = \begin{cases} 1, & 0 \le y_1 \le 1, \quad 0 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$$



if $y_1 \le 2y_2$ clearly $y_1 \le 3y_2$

$$P(Y_1 < 2Y_2 \mid Y_1 < 3Y_2) = \frac{P(Y_1 \le 2Y_2 \cap Y_1 \le 3Y_2)}{P(Y_1 \le 3Y_2)} = \frac{P(\underbrace{Y_1 \le 2Y_2}_{\frac{1}{2}y_1 \le y_2})}{P(\underbrace{Y_1 \le 3Y_2}_{\frac{1}{3}y_1 \le y_2})} = \frac{\int_0^1 \int_{\frac{y_1}{2}}^1 1 \, dy_2 \, dy_1}{\int_0^1 \int_{\frac{y_1}{3}}^1 1 \, dy_2 \, dy_1}$$

$$= \frac{\int_0^1 1 - \frac{y_1}{2} \, dy_1}{\int_0^1 (1 - \frac{y_1}{3}) \, dy_1} = \frac{3/4}{5/6} = \boxed{\frac{9}{10}}$$

Question: Let Y_1 and Y_2 be independent random variables that are both uniformly distributed on the interval $(0, 1)$. Find

$$P(Y_1 < 2Y_2 \mid Y_1 < 3Y_2).$$

$Y_1, Y_2 \sim \text{Uniform}(0, 1)$ and independent

$$P(Y_1 < 2Y_2 \mid Y_1 < 3Y_2) = \frac{P(Y_1 < 2Y_2 \text{ and } Y_1 < 3Y_2)}{P(Y_1 < 3Y_2)} = \frac{P(Y_1 < 2Y_2)}{P(Y_1 < 3Y_2)} = \frac{\int_0^1 \int_{y_1/2}^1 1 \, dy_2 \, dy_1}{\int_0^1 \int_{y_1/3}^1 1 \, dy_2 \, dy_1}$$

$y_2 = \frac{y_1}{2}$

$y_2 > \frac{y_1}{2}$

$y_2 = \frac{y_1}{3}$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

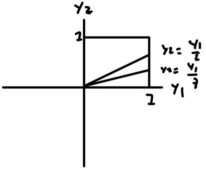
Recall: $Y \sim \text{unif}(a, b)$

$$f(y) = \begin{cases} \frac{1}{b-a}, & y \in (a, b) \\ 0, & \text{otherwise} \end{cases}$$

$$f_1(y_1) = \begin{cases} 1, & 0 \le y_1 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_2(y_2) = \begin{cases} 1, & 0 \le y_2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f(y_1, y_2) = \begin{cases} 1, & 0 \le y_1 \le 1, \ 0 \le y_2 \le 1 \\ 0, & \text{otherwise} \end{cases}$$



$$f(y_1, y_2) = \begin{cases} > 0, & a \le y_1 \le b, \ c \le y_2 \le d \\ 0, & \text{otherwise} \end{cases}$$

since independent, $f(y_1, y_2) = f_1(y_1) \cdot f_2(y_2)$

$f(y_1, y_2) = g(y_1) \cdot h(y_2)$

Question 5.81

Question: In Exercise 5.18, Y_1 and Y_2 denoted the lifetimes (in hundreds of hours) of components of types I and II. The joint density is

$$f(y_1, y_2) = \begin{cases} \frac{1}{8} y_1 e^{-(y_1+y_2)/2}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

One way to measure the relative efficiency of the two components is the ratio Y_2/Y_1.

Find $\mathbb{E}(Y_2/Y_1)$. *[Hint: In Exercise 5.61, it was proved that Y_1 and Y_2 are independent.]*

Y_1, Y_2 Independent $\Rightarrow E\left(\frac{Y_2}{Y_1}\right) = E(Y_2) \cdot E\left(\frac{1}{Y_1}\right)$

denominator

Gamma(2, 2) as $2^2 \Gamma(2) = 4 \cdot 1$

$f_1 = \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} \, dy_2 = \frac{1}{8} y_1 e^{-y_1/2} \int_0^\infty e^{-y_2/2} \, dy_2 = \frac{1}{4} y_1 e^{-y_1/2}, \quad y_1 > 0$

$E\left(\frac{1}{Y_1}\right) = E(Y_1^{-1}) = \frac{1}{2} \cdot \frac{\Gamma(1)}{\Gamma(2)} = \frac{1}{2}$

Recall: $E(Y^a) = \beta^a \frac{\Gamma(\alpha + a)}{\Gamma(\alpha)}$

u-sub then IBP

$f_2 = \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} \, dy_1 = \frac{1}{8} e^{-y_2/2} \underbrace{\int_0^\infty y_1 e^{-y_1/2} \, dy_1}_{4} = \frac{1}{2} e^{-y_2/2}$ Exponential with mean = 2

So $E(Y_2) = 2 \Rightarrow E(Y_2) \cdot E\left(\frac{1}{Y_1}\right) = 2$

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One way to measure the relative efficiency of the two components is the ratio Y_2/Y_1.

Find $\mathbb{E}(Y_2/Y_1)$. *[Hint: In Exercise 5.61, it was proved that Y_1 and Y_2 are independent.]*

Y_1, Y_2 are independent, $\Rightarrow E\left(\frac{Y_2}{Y_1}\right) = E(Y_2) \cdot E\left(\frac{1}{Y_1}\right) = Y_1^{-1}$

$u = -y_2/2$

$du = -\frac{1}{2} dy_2$

$$f_1(y_1) = \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_2 = \frac{1}{8} y_1 e^{-y_1/2} \int_0^\infty e^{-y_2/2} dy_2 = -\frac{1}{4} y_1 e^{-y_1/2} \int_0^{-\infty} e^u du = -\frac{1}{4} y_1 e^{-y_1/2} (e^u)_0^{-\infty}$$

$f_1(y_1) = \frac{1}{4} y_1 e^{-y_1/2}, \quad y_1 > 0$ gamma(2, 2) $\frac{\cdots}{\beta^\alpha \Gamma(\alpha)} = \frac{\cdots}{2^2 \underbrace{\Gamma(2)}_{1!}} = \frac{\cdots}{4}$

$f(y) = \left\{ \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^\alpha \Gamma(\alpha)} \right.$ $\quad E(Y^a) = \beta^a \frac{\Gamma(\alpha+a)}{\Gamma(\alpha)} \quad = E(Y_1^{-1}) = 2^{-1} \frac{\Gamma(1)}{\Gamma(2)} = \frac{1}{2} \cdot \frac{0! = 1}{1!} = \frac{1}{2}$

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$E(\frac{1}{y_1}) = \frac{1}{2}$

$E(y_2) = 2$

$\frac{1}{\beta} e^{-y/\beta}$, $E(y) = \beta$

Exponential $\beta = 2$

$$f_2(y_2) = \int_0^\infty \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_1 = \frac{1}{8} e^{-y_2/2} \underbrace{\int_0^\infty y_1 e^{-y_1/2} dy_1}_{=4} = \frac{1}{2} e^{-y_2/2} \Rightarrow E(y_2) = 2$$

$$E\left(\frac{y_2}{y_1}\right) = E\left(\frac{1}{y_1}\right) \cdot E(y_2) = \frac{1}{2} \cdot 2 = 1$$