ECO227Y5 Tutorial 12

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Question 5.136

Question: In Exercise 5.42, the number of defects per yard in a certain fabric, Y, was known to have a Poisson distribution with parameter λ. The parameter λ was assumed to be a random variable with density function

$$f(\lambda) = \begin{cases} e^{-\lambda}, & \lambda \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Exponential with $\theta = 1$

(a) Find the expected number of defects per yard by first finding the conditional expectation of Y given λ.

(b) Find the variance of Y.

(c) Is it likely that Y exceeds 9?

a.) $Y|\lambda \sim Poi(\lambda)$. Recall : Law of Iterated Expectation.

$E(Y_1) = E[E(Y_1|Y_2)]$

$E(Y) = E(E(Y|\lambda))$

$E(Y|\lambda) = \lambda$ since Poisson. So $E(\lambda) = 1$

b.) Recall: Law of Total variance $V(Y_1) = E(V(Y_1|Y_2)) + V(E(Y_1|Y_2))$

$V(Y) = E(\underbrace{V(Y|\lambda)}_{=\lambda}) + V(E(Y|\lambda))$

$V(Y) = E(\lambda) + V(\lambda) = 1 + 1^2 = 2$

c.) No, $\sigma = \sqrt{2}$, $E(Y) = 1$. 9 is many standard deviations away.

Question: Let Y_1 have an exponential distribution with mean λ, and let the conditional density of Y_2 given $Y_1 = y_1$ be

$$f(y_2 \mid y_1) = \begin{cases} \frac{1}{y_1}, & 0 \le y_2 \le y_1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $\mathbb{E}(Y_2)$ and $\mathbb{V}(Y_2)$, the unconditional mean and variance of Y_2.

$E(Y_2) = E(E(Y_2 \mid Y_1)) = E\left[\frac{Y_1}{2}\right] = \frac{E(Y_1)}{2} = \frac{\lambda}{2}$

$Y_1 \sim Exp(\lambda)$

$E(Y_2 \mid Y_1) = \int_0^{y_1} \frac{y_2}{y_1} dy_2 = \frac{y_2^2}{2y_1} \Big|_0^{y_1} = \frac{y_1^2}{2y_1} = \frac{y_1}{2}$

$V(Y_2 \mid Y_1 = y_1) = E(Y_2^2 \mid Y_1 = y_1) - [E(Y_2 \mid Y_1 = y_1)]^2$

$= \int_0^{y_1} \frac{y_2^2}{y_1} dy_2 - \frac{y_1^2}{4}$

$= \frac{y_2^3}{3y_1} \Big|_0^{y_1} - \frac{y_1^2}{4} = \frac{y_1^2}{3} - \frac{y_1^2}{4}$

$= \frac{y_1^2}{12}$

$V(Y_2) = E(V(Y_2 \mid Y_1)) + V(E(Y_2 \mid Y_1))$

$= E\left(\frac{Y_1^2}{12}\right) + V\left(\frac{Y_1}{2}\right)$

$= \frac{1}{12} E(Y_1^2) + \frac{V(Y_1)}{4}$

$= \frac{\lambda^2}{6} + \frac{\lambda^2}{4} = \frac{5\lambda^2}{12}$

$V(Y_1) = \lambda^2 = E(Y_1^2) - \lambda^2 \Rightarrow E(Y_1^2) = 2\lambda^2$

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Question 6.12

Question: Suppose that Y has a gamma distribution with parameters α and β and that $c > 0$ is a constant.

(a) Derive the density function of $U = cY$.

(b) Identify the density of U as one of the types we studied in Chapter 4. Be sure to identify any parameter values.

(c) The parameters α and β of a gamma-distributed random variable are, respectively, "shape" and "scale" parameters. How do the scale and shape parameters for U compare to those for Y?

a.) $Y \sim \text{Gamma}(\alpha, \beta)$, $U = cY$, $c \in \mathbb{R}$, $c > 0$.

By method of transformation.

Clearly cY is a monotone transformation. $u' = c$.

Theorem: If $U = h(Y)$, where h is a monotone function and Y is a random variable. $f_U(u) = f_Y(h^{-1}(u)) \left| \frac{dh^{-1}(u)}{du} \right|$

Recall: $Y \sim \text{Gamma}(\alpha, \beta)$. $f(y) = \frac{y^{\alpha-1} e^{-y/\beta}}{\Gamma(\alpha)\beta^\alpha}$

$u = h(y) = cy \Rightarrow \frac{u}{c} = y = h^{-1}(u)$

$\frac{dh^{-1}(u)}{du} = \frac{1}{c} \Rightarrow f_U(u) = \frac{\left(\frac{u}{c}\right)^{\alpha-1} e^{-\frac{u}{c\beta}}}{\Gamma(\alpha)\beta^\alpha} \cdot \frac{1}{c} = \frac{u^{\alpha-1} e^{-u/c\beta}}{\Gamma(\alpha)(c\beta)^\alpha}$, $u > 0$

b.) $U \sim \text{Gamma}(\alpha, c\beta)$

c.) shape stays the same: α

scale multiplies by c: $\beta \to c\beta$

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Extra

Question: Suppose the English test scores of students in elementary schools follow a normal distribution with mean μ and variance θ. Suppose you observe test scores of students from all elementary schools in Ontario. Since schools in Ontario have different teacher qualities or student backgrounds, you believe that the mean test scores from different schools are different, and it is reasonable to assume that μ also follows a normal distribution with mean 70 and variance 20, and that θ follows a gamma distribution, $\mathrm{Gamma}(\mu, 2\mu)$. *[Hint: both μ and θ are random variables and θ depends on μ in a particular way.]*

(a) What is the mean of English test scores for all elementary schools in Ontario?

(b) What is the variance of English test scores for all elementary schools in Ontario?

a.) scores $Y \mid (\mu, \theta) \sim N(\mu, \theta)$

$\mu \sim N(70, 20)$

$\theta \mid \mu \sim \text{Gamma}(\mu, 2\mu)$

$E(Y \mid \mu, \theta) = \mu$

$E(Y) = E(E(Y \mid \mu, \theta)) = E(\mu) = 70$

b.) $V(Y) = E(V(Y \mid \mu, \theta)) + V(E(Y \mid \mu, \theta))$

$V(Y) = E(\theta) + V(\mu)$, $V(Y \mid \mu, \theta) = \theta$

$V(Y) = 9840 + 20 = 9860$

from Gamma

$E(\theta) = E(E(\theta \mid \mu)) = E(2\mu^2) = 2 \cdot 4920 = 9840$

$V(\mu) = E(\mu^2) - E(\mu)^2 \Rightarrow 20 = E(\mu^2) - E(\mu)^2$

$20 = E(\mu^2) - 70^2$

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