ECO227Y5 Tutorial 13

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Question 6.23

Question: In Exercise 6.1, we considered a random variable Y with probability density function

$$f(y) = \begin{cases} 2(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere}, \end{cases}$$

and used the method of distribution functions to find the density functions of:

$$U_1 = 2Y - 1, \qquad U_3 = Y^2.$$

Use the method of transformation to find the densities of U_1 and U_3.

method of Distribution Functions:

$U_1 = 2Y - 1$

$P(U_1 \le u) = P(2Y - 1 \le u) = P\left(Y \le \frac{u+1}{2}\right)$. Find $F(y)$

$$F(y) = \int_0^y 2(1-y)\,dy = \begin{cases} 0, & y < 0 \\ 2y - y^2, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

$y \in [0,1] \Rightarrow u_1 \in [-1, 1]$

$$F\left(\frac{u+1}{2}\right) = u + 1 - \frac{u^2 + 2u + 1}{4} = \frac{4u + 4 - u^2 - 2u - 1}{4}$$

$$F\left(\frac{u+1}{2}\right) = \frac{-u^2 + 2u + 3}{4} = F_1(u_1)$$

Differentiate

so $f_1(u_1) = \begin{cases} \frac{1 - u_1}{2}, & u_1 \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$

$U_3 = Y^2$: $P(U_3 \le u) = P(Y^2 \le u)$

$y \in [0,1] \Rightarrow P(-\sqrt{u} \le Y \le \sqrt{u})$

$\Rightarrow u_3 \in [0,1) \Rightarrow F(\sqrt{u}) - F(-\sqrt{u})$

$F(\sqrt{u}) = 2\sqrt{u} - u \Rightarrow F(\sqrt{u}) - 0 = 2\sqrt{u} - u = F(u_3)$

$F(-\sqrt{u}) = 0$ as $\underline{Y \ge 0}$

Differentiate

so $f_3(u_3) = \begin{cases} \frac{1}{\sqrt{u_3}} - 1, & u_3 \in (0, 1] \\ 0, & \text{otherwise} \end{cases}$

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and used the method of distribution functions to find the density functions of:

Theorem:

If $U = h(Y)$, where h is a monotone function and Y is a random variable. $f_U(u) = f_Y(h^{-1}(u)) \left| \frac{d h^{-1}(u)}{du} \right|$

$$U_1 = 2Y - 1, \qquad U_3 = Y^2.$$

Use the method of transformation to find the densities of U_1 and U_3.

Method of Transformations:

$u_1 = 2Y - 1$. Check if monotone: $\frac{du_1}{dY} = 2$ ✓

isolate for Y

$u_1 = 2Y - 1 \Rightarrow Y = \frac{u_1 + 1}{2}$, $\frac{dY}{du_1} = \frac{1}{2}$

$f_1(u_1) = 2\left(1 - \frac{u_1+1}{2}\right) \cdot \left|\frac{1}{2}\right|$

$f_1(u_1) = 1 - \frac{u_1+1}{2} = \frac{1-u_1}{2}$ $\quad Y \in [0,1] \Rightarrow u \in [-1, 1]$

$$f_1(u_1) = \begin{cases} \frac{1-u_1}{2}, & u \in [-1,1] \\ 0, & \text{otherwise} \end{cases}$$

$u_3 = Y^2$, $\frac{du_3}{dY} = 2Y$, on $Y \in (0,1]$ it is monotonic.

$u_3 = Y^2 \Rightarrow Y = \sqrt{u_3}$ only positive as $Y \in [0,1]$

$\frac{dY}{du_3} = \frac{1}{2\sqrt{u_3}}$

$f_3(u_3) = 2(1 - \sqrt{u_3}) \left| \frac{1}{2\sqrt{u_3}} \right| = \frac{1-\sqrt{u_3}}{\sqrt{u_3}} = \frac{1}{\sqrt{u_3}} - 1$

$$f_3(u_3) = \begin{cases} \frac{1}{\sqrt{u_3}} - 1, & u_3 \in (0,1] \\ 0, & \text{otherwise} \end{cases}$$

Question 6.32

Question: In Exercise 6.5, we considered a random variable Y that has a uniform distribution on the interval $[1, 5]$. The cost of delay is given by

$Y \sim Uni(1, 5) \Rightarrow f(y) = \begin{cases} \frac{1}{4}, & y \in [1,5] \\ 0, & \text{elsewhere} \end{cases}$

$$U = 2Y^2 + 3.$$

Use the method of transformations to derive the density function of U.

$u = 2Y^2 + 3$ check monotonicity: $\frac{du}{dY} = 4Y$, since $Y \in [1,5]$

$u = 2Y^2 + 3 \Rightarrow 2Y^2 = u - 3 \Rightarrow Y = \underbrace{\sqrt{\frac{u-3}{2}}}_{= h^{-1}(u)}$ only + as $Y \in [1,5]$

If $Y = 1 \Rightarrow u = 5$

$Y = 5 \Rightarrow u = 53$

$\frac{dY}{du} = \frac{1}{4\sqrt{\frac{u-3}{2}}}$

$f_U(u) = \frac{1}{4} \cdot \left| \frac{1}{4\sqrt{\frac{u-3}{2}}} \right| = \frac{1}{16\sqrt{\frac{u-3}{2}}} \Rightarrow f_U(u) = \begin{cases} \frac{1}{16\sqrt{\frac{u-3}{2}}}, & u \in [5, 53] \\ 0 & \end{cases}$

Question 6.34

Question: A density function sometimes used by engineers to model lengths of life of electronic components is the Rayleigh density, given by

– we had $t^{1/2}$ so $1/2 = \alpha - 1 \ni \alpha = \frac{3}{2}$

Definition: For any $\alpha > 0$: $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\,dt$

Properties: 1. $\Gamma(n+1) = n\Gamma(n)$
2. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$f(y) = \begin{cases} \frac{2y}{\theta} e^{-y^2/\theta}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) If Y has the Rayleigh density, find the probability density function for $U = Y^2$.

(b) Use the result of part (a) to find $\mathbb{E}(Y)$ and $\mathbb{V}(Y)$.

$y > 0 \ni y^2 = u > 0$

a.) $u = Y^2$, check monotonicity: $\frac{du}{dY} = 2Y$, $Y > 0 \ni$ monotone

$Y = \sqrt{u} = h^{-1}(u)$, + as $Y > 0$

$\frac{dY}{du} = \frac{1}{2\sqrt{u}} \ni f_U(u) = \frac{2\sqrt{u}}{\theta} e^{-u/\theta} \cdot \frac{1}{2\sqrt{u}} = \frac{e^{-u/\theta}}{\theta}$

$\ni f_U(u) = \begin{cases} \frac{e^{-u/\theta}}{\theta}, & u > 0 \\ 0, & \text{elsewhere} \end{cases}$

b.) Notice we have an exponential! Recall: $Y \sim Exp(\theta) \ni f(y) = \frac{1}{\theta} e^{-y/\theta}$, $\theta > 0$, $y > 0$

$E(Y^2) = E(u) = \theta$

let $t = u/\theta$, $dt = \frac{1}{\theta} du$ note: * is the gamma integral $\int_0^\infty t^{1/2} e^{-t} dt = \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$

$E(Y) = E(\sqrt{u}) = \int_0^\infty \sqrt{u} \frac{e^{-u/\theta}}{\theta} du = \sqrt{\theta} \int_0^\infty t^{1/2} e^{-t} dt = \frac{\sqrt{\theta\pi}}{2} \ni V(Y) = \theta - \frac{\theta\pi}{4} = \theta(1 - \frac{\pi}{4})$

$\sqrt{u} = \sqrt{t\theta}$

Recall: $Y \sim \text{Gamma}(\alpha, \beta)$

$$f(y) = \frac{y^{\alpha-1} e^{-y/\beta}}{\Gamma(\alpha)\,\beta^{\alpha}}$$

we want a PDF with form $f(y) = y^{\alpha-1} e^{-y/\beta}$

Need PDF to integrate to 1: so $\int_0^{\infty} c\, y^{\alpha-1} e^{-y/\beta}\, dy = 1$, $c \in \mathbb{R}$

$\int_0^{\infty} y^{\alpha-1} e^{-y/\beta}\, dy$, let $t = \frac{y}{\beta}$ $\Rightarrow y = \beta t$, $\frac{dt}{dy} = \frac{1}{\beta} \Rightarrow dt = \frac{dy}{\beta}$

$$\Rightarrow \int_0^{\infty} (\beta t)^{\alpha-1} e^{-t} \cdot \beta\, dt = \int_0^{\infty} \beta^{\alpha} t^{\alpha-1} e^{-t}\, dt = \beta^{\alpha} \int_0^{\infty} t^{\alpha-1} e^{-t}\, dt = \beta^{\alpha}\Gamma(\alpha)\text{, a constant.}$$

Recall Definition: $\int_0^{\infty} t^{\alpha-1} e^{-t}\, dt = \Gamma(\alpha)$

so $\int_0^{\infty} y^{\alpha-1} e^{-y/\beta}\, dy = \beta^{\alpha}\Gamma(\alpha)$ so if $c = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \Rightarrow \int_0^{\infty} c \cdot y^{\alpha-1} e^{-y/\beta}\, dy = \int_0^{\infty} \frac{y^{\alpha-1} e^{-y/\beta}}{\beta^{\alpha}\Gamma(\alpha)}\, dy$

$$\Rightarrow \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \cdot \int_0^{\infty} y^{\alpha-1} e^{-y/\beta} = \frac{\beta^{\alpha}\Gamma(\alpha)}{\beta^{\alpha}\Gamma(\alpha)} = 1$$