ECO227Y5 Tutorial 14

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Question: Let $Y_1, Y_2, \ldots, Y_n$ be independent, uniformly distributed random variables on the interval $[0, \theta]$. Find the

$Y_i \sim uni[0,\theta]$ $f_{Y_i}(y) = \begin{cases} \frac{1}{\theta}, & y \in (0,\theta] \\ 0, & \text{otherwise} \end{cases}$

(a) probability distribution function of $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$.

(b) density function of $Y_{(n)}$. $F(y_i) = \frac{y}{\theta}$

(c) mean and variance of $Y_{(n)}$.

Recall: $Y_1, \ldots, Y_n$ iid.

Sort them from smallest to largest:

$Y_{(1)} \le Y_{(2)} \le \cdots \le Y_{(n)}$

$Y_{(1)} = \min(Y_1, \ldots, Y_n)$

$Y_{(n)} = \max(Y_1, \ldots, Y_n)$

$Y_{(k)} =$ k th smallest observation

a.) $Y_{(n)} = \max(Y_1, \ldots Y_n)$

$F_{Y_{(n)}}(y) = P(Y_{(n)} \le y) \ni P(Y_1 \le y, \ldots, Y_n \le y)$

if the largest Y is less than y then all y's must be less than y as well.

$\ni \underbrace{P(Y_1 \le y)}_{F_1(y)} \cdot \underbrace{P(Y_2 \le y)}_{F_2(y)} \cdots \underbrace{P(Y_n \le y)}_{F_n(y)}$ as independent

All same

$F_i(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{\theta}, & y \in (0, \theta) \\ 1, & y > \theta \end{cases} \ni P(Y_{(n)} \le y) = (F(y))^n = \left(\frac{y}{\theta}\right)^n$

So $F_{Y_{(n)}}(y) = \begin{cases} 0, & y < 0 \\ \left(\frac{y}{\theta}\right)^n, & y \in (0, \theta] \\ 1, & y > \theta \end{cases}$

b.) $f_{Y_{(n)}}(y) = \frac{d F_{Y_{(n)}}(y)}{dy} = n\left(\frac{y}{\theta}\right)^{n-1} \cdot \frac{1}{\theta} = \frac{n\left(\frac{y}{\theta}\right)^{n-1}}{\theta} = \frac{n y^{n-1}}{\theta^n}$

$f_{Y_{(n)}}(y) = \begin{cases} \frac{n y^{n-1}}{\theta^n}, & y \in [0, \theta] \\ 0, & \text{otherwise} \end{cases}$

c.) $E_{Y_{(n)}}(Y) = \int_0^\theta \frac{n y^n}{\theta^n} dy = \frac{n y^{n+1}}{(n+1)\theta^n} \Big|_0^\theta$

$= \frac{n \theta^{n+1}}{(n+1)\theta^n} = \frac{n\theta}{n+1}$

$V_{Y_{(n)}}(Y) = E_{Y_{(n)}}(Y^2) - E_{Y_{(n)}}(Y)^2$

$= \int_0^\theta \frac{n y^{n+1}}{\theta^n} dy = \frac{n y^{n+2}}{(n+2)\theta^n}\Big|_0^\theta - \left(\frac{n\theta}{n+1}\right)^2$

$= \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2} = \frac{n\theta^2}{(n+2)(n+1)^2}$

Question 6.75

Question: Refer to Exercise 6.74. Suppose that the number of minutes you need to wait for a bus is uniformly distributed on the interval $[0, 15]$. If you take the bus five times, what is the probability that your longest wait is less than 10 minutes?

Let Y_i denote the time you wait $Y \sim uni(0, 15]$, each i being each time you wait. $i \in \{1,2,3,4,5\}$

The question translates to $P(Y_{(5)} < 10) = F_{Y(5)}(10)$

Recall: $F_{Y(n)}(y) = \begin{cases} 0, & y<0 \\ \left(\frac{y}{\theta}\right)^n, & y \in (0,\theta] \\ 1, & y > \theta \end{cases}$, $n=5$, $\theta = 15$ so $F_{Y(5)}(y) = \begin{cases} 0, & y<0 \\ \left(\frac{y}{15}\right)^5, & y \in (0,15] \\ 1, & y>15 \end{cases}$

$F_{Y(5)}(10) = \left(\frac{10}{15}\right)^5 = 0.1317$

Question 7.13

Question: The Environmental Protection Agency is concerned with the problem of setting criteria for the amounts of certain toxic chemicals to be allowed in freshwater lakes and rivers. A common measure of toxicity for any pollutant is the concentration of the pollutant that will kill half of the test species in a given amount of time (usually 96 hours for fish species). This measure is called LC50 (lethal concentration killing 50% of the test species). In many studies, the values contained in the natural logarithm of LC50 measurements are normally distributed, and, hence, the analysis is based on $\ln(\mathrm{LC50})$ data.

Studies of the effects of copper on a certain species of fish show the variance of $\ln(\mathrm{LC50})$ measurements to be around 0.4 (with concentration measurements in milligrams per liter). If $n = 10$ studies on LC50 for copper are to be completed, find the probability that the sample mean of $\ln(\mathrm{LC50})$ will differ from the true population mean by no more than 0.5.

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$n = 10, \ \sigma^2 = 0.4$

$\ln(LC50) = Y \sim N(\mu, 0.4)$

need to convert to z

$$P(|\bar{Y} - \mu| \le 0.5) \Rightarrow P(-0.5 \le \bar{Y} - \mu \le 0.5) = P\left(\frac{-0.5}{\sqrt{\frac{0.4}{10}}} \le \overbrace{\frac{\bar{Y}-\mu}{\sqrt{\frac{\sigma^2}{n}}}}^{\approx z} \le \frac{0.5}{\sqrt{\frac{0.4}{10}}}\right)$$

$$\Rightarrow P(-2.5 \le z \le 2.5) = 2(P(z \ge 0) - P(z \ge 2.5)) = 2 \cdot (0.5 - 0.0062) = 0.9876$$

Recall: For $Y \sim N(\mu, \sigma^2)$

$$z = \frac{Y - \mu}{\sqrt{\sigma^2}}$$

For $\bar{Y}$: $\bar{Y} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

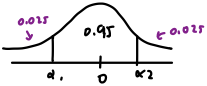
so $SE = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$

Question 7.14

Question: If in Exercise 7.13 we want the sample mean of ln$(\mathrm{LC50})$ to differ from the population mean by no more than 0.5 with probability 0.95, how many tests should be run?

$P(|\bar{Y}-\mu| \le 0.5) = 0.95$, can change n.

$$\Rightarrow P\left(\frac{-0.5}{\frac{\sqrt{0.4}}{\sqrt{n}}} \le \underbrace{\frac{\bar{Y}-\mu}{\frac{\sqrt{0.4}}{\sqrt{n}}}}_{=z} \le \frac{0.5}{\frac{\sqrt{0.4}}{\sqrt{n}}} \right) = 0.95 \Rightarrow P\left(\underbrace{\frac{-0.5}{\frac{\sqrt{0.4}}{\sqrt{n}}}}_{\alpha_1} \le z \le \underbrace{\frac{0.5}{\frac{\sqrt{0.4}}{\sqrt{n}}}}_{\alpha_2} \right) = 0.95$$



$P(\alpha_1 \le z \le \alpha_2) = 0.95 \Rightarrow$

where $\alpha_1 = -\alpha_2$.

so $P(z > \alpha_2) = P(z < \alpha_1) = 0.025$

By checking table, $\alpha_2 = 1.96 \Rightarrow \alpha_1 = -1.96$

$$\alpha_2 = 1.96 = \frac{0.5}{\frac{\sqrt{0.4}}{\sqrt{n}}} \Rightarrow 1.96 = \frac{0.5\sqrt{n}}{\sqrt{0.4}} \Rightarrow n = 6.14656 \approx 7$$ rounded up

Question 7.15

Question: Suppose that $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ are independent random samples, with the variables X_i normally distributed with mean μ_1 and variance σ_1^2 and the variables Y_i normally distributed with mean μ_2 and variance σ_2^2. The difference between the sample means, $\overline{X} - \overline{Y}$, is then a linear combination of $m + n$ normally distributed random variables and, by Theorem 6.3, is itself normally distributed.

$x_1, \ldots, x_m \overset{iid}{\sim} N(\mu_1, \sigma_1^2)$, $Y_1, \ldots, Y_n \overset{iid}{\sim} N(\mu_2, \sigma_2^2)$

$\Rightarrow \bar{X} \sim N(\mu_1, \frac{\sigma_1^2}{m})$ $\quad \Rightarrow \bar{Y} \sim N(\mu_2, \frac{\sigma_2^2}{n})$

(a) Find $\mathbb{E}(\overline{X} - \overline{Y})$. $E(\bar{x} - \bar{y}) = E(\bar{x}) - E(\bar{y}) = \mu_1 - \mu_2$

$v(\bar{x} - \bar{y}) = v(\bar{x}) - 2\,cov(\bar{x}, \bar{y}) + v(\bar{y})$

$(\bar{x} - \bar{y})^2 = \bar{x}^2 - 2\bar{x}\bar{y} + \bar{y}^2$

(b) Find $\mathbb{V}(\overline{X} - \overline{Y})$. $v(\bar{x} - \bar{y}) = v(\bar{x}) + v(\bar{y}) = \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}$

(c) Suppose that $\sigma_1^2 = 2$, $\sigma_2^2 = 2.5$, and $m = n$. Find the sample sizes so that $(\overline{X} - \overline{Y})$ will be within 1 unit of $(\mu_1 - \mu_2)$ with probability 0.95.

so $\bar{x} - \bar{y} \sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n})$

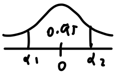
$\bar{x} - \bar{y} \sim N(\mu_1 - \mu_2, \frac{4.5}{n})$

$P(|\underbrace{(\bar{x} - \bar{y})}_{\text{"Y"}} - \underbrace{(\mu_1 - \mu_2)}_{\text{"}\mu\text{"}}| \le 1) = 0.95$

$\Rightarrow P(-1 \le (\bar{x} - \bar{y}) - (\mu_1 - \mu_2) \le 1) = 0.95 \Rightarrow P\left(\overset{\alpha_1}{\frac{-1}{\frac{\sqrt{4.5}}{\sqrt{n}}}} \le Z \le \overset{\alpha_2}{\frac{1}{\frac{\sqrt{4.5}}{\sqrt{n}}}}\right) = 0.95$

From last Question

$\Rightarrow \alpha_2 = 1.96 \Rightarrow \frac{\sqrt{n}}{\sqrt{4.5}} = 1.96 \Rightarrow n = 17.2872 \approx 18$



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