ECO227Y5 Tutorial 16

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Question 8.36

Question: If $Y_1, Y_2, \ldots, Y_n$ denote a random sample from an exponential distribution with mean θ, then $E(Y_i) = \theta$ and $V(Y_i) = \theta^2$. Thus, $E(\bar{Y}) = \theta$ and $V(\bar{Y}) = \theta^2/n$, or $\sigma_{\bar{Y}} = \theta/\sqrt{n}$. Suggest an unbiased estimator for θ and provide an estimate for the standard error of your estimator.

$Y_1, \ldots, Y_n \overset{iid}{\sim} \text{Exp}(\theta)$

Recall: $\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$

$E(\bar{Y}) = \theta$ so clearly $\bar{Y}$ is an unbiased estimator for θ

The standard error is the standard deviation of an estimator's sampling distribution.

so since $\bar{Y}$ is an unbiased estimator, an estimator for $\sigma_{\bar{Y}} = \frac{\theta}{\sqrt{n}}$ is $\frac{\bar{Y}}{\sqrt{n}}$

Question: Suppose that the random variable Y has a gamma distribution with parameters $\alpha = 2$ and an unknown β. In Exercise 6.46, you used the method of moment-generating functions to prove a general result implying that $2Y/\beta$ has a χ^2 distribution with 4 degrees of freedom (df). Using $2Y/\beta$ as a pivotal quantity, derive a 90% confidence interval for β. $Y \sim \text{Gamma}(2, \beta)$

Ex. $\frac{Y-\mu}{\sigma} \sim N(0,1)$

note: Pivotal quantity is a function of the data and the unknown parameter.

Given that $\frac{2Y}{\beta} \sim \chi^2_4$ we find a, b such that $P(a \le \frac{2Y}{\beta} \le b) = 0.90$

Good practice to place equal weights on tails

$\Rightarrow P(a \le \chi^2_4 \le b) = 0.90$

$\Rightarrow P(\chi^2_4 \ge a) - P(\chi^2_4 \ge b) = 0.90$

$\qquad\qquad\qquad\qquad\quad 0.05$

$\Rightarrow P(\chi^2_4 \ge a) = 0.95 \Rightarrow a = 0.710721$

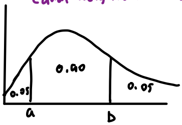
$\Rightarrow P(\chi^2_4 \ge b) = 0.05 \Rightarrow b = 9.48773$

$\Rightarrow P(0.710721 \le \frac{2Y}{\beta} \le 9.48773)$

solve for β

$\Rightarrow P\left(\frac{0.710721}{2Y} \le \frac{1}{\beta} \le \frac{9.48773}{2Y}\right)$

$\Rightarrow P\left(\frac{2Y}{9.48773} \le \beta \le \frac{2Y}{0.710721}\right)$



why flip? Ex. $4 < 10$ but $\frac{1}{4} > \frac{1}{10}$

so 90% CI for β: $\left[\frac{2Y}{9.48773}, \frac{2Y}{0.710721}\right]$

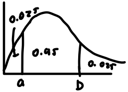
Question 8.41

Question: Suppose that Y is normally distributed with mean 0 and unknown variance σ^2. Then Y^2/σ^2 has a χ^2 distribution with 1 df. Use the pivotal quantity Y^2/σ^2 to find a $Y \sim N(0, \sigma^2) \Rightarrow \frac{Y^2}{\sigma^2} \sim \chi^2_1$

(a) 95% confidence interval for σ^2.

(b) 95% <u>upper confidence</u> limit for σ^2.   (one sided)

(c) 95% lower confidence limit for σ^2.



a.) $P(a \le \frac{Y^2}{\sigma^2} \le b) = 0.95 \Rightarrow P(a \le \chi^2_1 \le b) = 0.95$ -0.025

$\Rightarrow P(\chi^2_1 \ge a) - P(\chi^2_1 \ge b) = 0.95$

$\Rightarrow P(\chi^2_1 \ge a) = 0.975 \Rightarrow a = 0.0009821$

and $P(\chi^2_1 \ge b) = 0.025 \Rightarrow b = 5.02389$

$P(0.0009821 \le \frac{Y^2}{\sigma^2} \le 5.02389) \Rightarrow P(\frac{Y^2}{5.02389} \le \sigma^2 \le \frac{Y^2}{0.0009821}) \Rightarrow$ 95% CI for σ^2: $\left[\frac{Y^2}{5.02389}, \frac{Y^2}{0.0009821}\right]$

* since σ^2 in denom we use ?

b.) $P(\frac{Y^2}{\sigma^2} \ge c) = 0.95 \Rightarrow P(\chi^2_1 \ge c) = 0.95 \Rightarrow c = 0.0039321 \Rightarrow P(\frac{Y^2}{\sigma^2} \ge 0.0039321) = 0.95 \Rightarrow P(\sigma^2 \le \frac{Y^2}{0.0039321})$

so 95% upper limit C.I. for σ^2: $\left(0, \frac{Y^2}{0.0039321}\right]$

variance can't be negative

b.)

c.)

c.) $P(\frac{Y^2}{\sigma^2} \le d) = 0.95 \Rightarrow P(\chi^2_1 > d) = 0.05 \Rightarrow d = 3.84146$

so $P(\frac{Y^2}{\sigma^2} \le 3.84146) \Rightarrow P(\sigma^2 \ge \frac{Y^2}{3.84146})$ so 95% lower limit C.I. for σ^2: $\left[\frac{Y^2}{3.84146}, \infty\right)$

Question 8.43

Question: Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample of size n from a population with a uniform distribution on the interval $(0, \theta)$. Let $Y_{(n)} = \max(Y_1, Y_2, \ldots, Y_n)$ and $U = \frac{1}{\theta} Y_{(n)}$.

$Y_i \sim \text{uni}(0,\theta)$

$f(y) = \frac{1}{\theta}$ so $F(y) = \frac{y}{\theta}$

on $(0,\theta)$

(a) Show that U has distribution function

$Y_i \overset{iid}{\sim} \text{uni}[0,\theta]$, $u = \frac{1}{\theta} Y_{(n)}$

$P(u \le u) = P(Y_{(n)} \le u\theta) = P(Y_i \le u\theta)^n$

$\Rightarrow F(u\theta)^n = \left(\frac{u\theta}{\theta}\right)^n = u^n$

If $u = 1 \Rightarrow 1 = \frac{1}{\theta} Y_{(n)} \Rightarrow Y_{(n)} = \theta$

so max.

If $u = 0 \Rightarrow 0 = \frac{1}{\theta} Y_{(n)} \Rightarrow Y_{(n)} = 0$

so min

$$F_U(u) = \begin{cases} 0, & u < 0, \\ u^n, & 0 \le u \le 1, \\ 1, & u > 1. \end{cases}$$

(b) Because the distribution of U does not depend on θ, U is a pivotal quantity. Find a 95% lower confidence bound for θ.

$P(\frac{1}{\theta} Y_{(n)} \le a) = 0.95 \Rightarrow P(u < a) = 0.95 \Rightarrow F(a) = 0.95 \Rightarrow a^n = 0.95 \Rightarrow a = 0.95^{1/n}$

$P(\frac{1}{\theta} Y_{(n)} \le 0.95^{1/n}) = 0.95 \Rightarrow P\left(\theta \ge \frac{Y_{(n)}}{0.95^{1/n}}\right) = 0.95$ so 95% lower limit C.I. for θ: $\left[\frac{Y_{(n)}}{0.95^{1/n}}, \infty\right)$

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Question 8.47

Question: Refer to Exercise 8.46. Assume that $Y_1, Y_2, \ldots, Y_n$ is a sample of size n from an exponential distribution with mean θ.

(a) Use the method of moment-generating functions to show that $\dfrac{2\sum_{i=1}^{n} Y_i}{\theta}$ is a pivotal quantity and has a χ^2 distribution with $2n$ df.

a.) I will call $\dfrac{2\sum_{i=1}^{n} Y_i}{\theta} = X$. $m_X(t) = E(e^{Xt}) = E\left(e^{\frac{2\sum_{i=1}^{n} Y_i t}{\theta}}\right) = E\left(e^{2t\left(\frac{Y_1}{\theta} + \frac{Y_2}{\theta} + \cdots + \frac{Y_n}{\theta}\right)}\right)$

$= E\left(e^{\frac{2t}{\theta}Y_1} \cdot e^{\frac{2t}{\theta}Y_2} \cdots e^{\frac{2t}{\theta}Y_n}\right)$, Y_i are iid so,

$= E\left(e^{\frac{2t}{\theta}Y_i}\right)^n = \left(m_{Y_i}\left(\frac{2t}{\theta}\right)\right)^n$ where $i \in \{1, \ldots, n\}$

$\Rightarrow \left(\left(1 - \theta \cdot \frac{2t}{\theta}\right)^{-1}\right)^n = (1-2t)^{-n}$

so this is the MGF for χ^2_{2n}.

Recall: For $Y \sim Exp(\theta)$, $m_Y(t) = (1-\theta t)^{-1}$

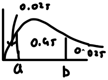
Recall: $Y \sim \chi^2_v$, $m_Y(t) = (1-2t)^{-v/2}$

Question 8.47

Question: Refer to Exercise 8.46. Assume that $Y_1, Y_2, \ldots, Y_n$ is a sample of size n from an exponential distribution with mean θ.

(b) Use the pivotal quantity $\dfrac{2\sum_{i=1}^{n} Y_i}{\theta}$ to derive a 95% confidence interval for θ.

(c) If a sample of size $n = 7$ yields $\bar{y} = 4.77$, use the result from part (b) to give a 95% confidence interval for θ.



b) $P(a \le x \le b) = 0.95 \Rightarrow P(a \le \chi^2_{2n} \le b) = 0.95$

call $a = \chi^2_{2n,0.975}$, $b = \chi^2_{2n,0.025}$

$$P\left(\chi^2_{2n,0.975} \le \frac{2\sum_{i=1}^{n} Y_i}{\theta} \le \chi^2_{2n,0.025}\right)$$

$$\Rightarrow P\left(\frac{2\sum_{i=1}^{n} Y_i}{\chi^2_{2n,0.025}} \le \theta \le \frac{2\sum_{i=1}^{n} Y_i}{\chi^2_{2n,0.975}}\right) = 0.95$$

so 95% C.I. for θ: $\left[\dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{2n,0.025}}, \dfrac{2\sum_{i=1}^{n} Y_i}{\chi^2_{2n,0.975}}\right]$

c.) $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i \Rightarrow 4.77 = \frac{1}{7}\sum_{i=1}^{7} y_i \Rightarrow \sum_{i=1}^{7} y_i = 33.39$

and $d.f = 2 \cdot 7 = 14$. $\chi^2_{14,0.025} = 26.1190$, $\chi^2_{14,0.975} = 5.62872$

so 95% C.I. for θ: $\left[\dfrac{2 \cdot 33.39}{26.1190}, \dfrac{2 \cdot 33.39}{5.62872}\right] = [2.557, 11.864]$

Question 8.59

Question: When it comes to advertising, "tweens" are not ready for the hard-line messages that advertisers often use to reach teenagers. The Geppeto Group study found that 78% of "tweens" understand and enjoy ads that are silly in nature. Suppose that the study involved $n = 1030$ "tweens".

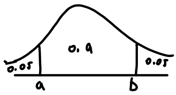
(a) Construct a 90% confidence interval for the proportion of "tweens" who understand and enjoy ads that are silly in nature.

(b) Do you think that "more than 75%" of all "tweens" enjoy ads that are silly in nature? Why?

a.) $Y \sim Bin(n, p)$, unbiased estimator for p is $\hat{p} = \frac{Y}{n}$ as $E(\hat{p}) = E\left(\frac{Y}{n}\right) = \frac{1}{n} E(Y) = \frac{1}{n} \cdot np = p$

$\hat{p} = 0.78$

$\sigma^2_{\hat{p}} = \frac{\hat{p}(1-\hat{p})}{n} = \frac{0.78 \cdot 0.22}{1030} = 1.666 \cdot 10^{-4} \Rightarrow \sigma_{\hat{p}} = \sqrt{1.666 \cdot 10^{-4}}$



$P\left(a \le \frac{\hat{p} - p}{\sigma_{\hat{p}}} \le b\right) = 0.9$, can use z from CLT. $\Rightarrow a = -1.645$, $b = 1.645$

$\Rightarrow P(-1.645 \cdot \sigma_{\hat{p}} - \hat{p} \le -p \le 1.645 \cdot \sigma_{\hat{p}} - \hat{p}) = 0.9 \Rightarrow P(-1.645\sigma_{\hat{p}} + \hat{p} \le p \le 1.645\sigma_{\hat{p}} + \hat{p})$

$\Rightarrow$ 90% C.I. for p: $[0.7588, 0.80127]$

b. The 90% C.I is above 0.75, from around 0.76 to 0.80 so it is reasonable to say that more than 75% of all tweens enjoy silly ads.