

ECO227Y5 Tutorial 19

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Question 9.25

Question: Suppose that Y_1, Y_2, \dots, Y_n denote a random sample of size n from a normal distribution with mean μ and variance 1. Consider the first observation Y_1 as an estimator for μ .

- (a) Show that Y_1 is an unbiased estimator for μ .
- (b) Find $P(|Y_1 - \mu| \leq 1)$.
- (c) Look at the basic definition of consistency given in Definition 9.2. Based on the result of part (b), is Y_1 a consistent estimator for μ ?

a.) $E(Y_1) = \mu$ clearly as $Y_1 \sim N(\mu, 1)$ is unbiased

b.) $P(|Y_1 - \mu| \leq 1) = P(-1 \leq Y_1 - \mu \leq 1) = P\left(-\frac{1}{1} \leq Z \leq \frac{1}{1}\right) = P(-1 \leq Z \leq 1) = 0.6817$

c.) No, consistency requires that $P(|Y_1 - \mu| > \epsilon) \rightarrow 0$ as $n \rightarrow \infty$ for all $\epsilon > 0$.

But $P(|Y_1 - \mu| \leq 1) \neq P(|Y_1 - \mu| > 1) = 0.3183$ does not approach 0 as $n \rightarrow \infty$

Question 9.30

Question: Let Y_1, Y_2, \dots, Y_n be independent random variables, each with probability density function

$$f(y) = \begin{cases} 3y^2, & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that \bar{Y} converges in probability to some constant and find the constant.

$$E(\bar{Y}) = E\left(\frac{1}{n} \sum_{i=1}^n Y_i\right) = \frac{1}{n} \sum_{i=1}^n E(Y_i) = \frac{3}{4}$$

$$E(Y_i) = \int_0^1 3y^3 dy = \frac{3y^4}{4} \Big|_0^1 = \frac{3}{4}$$

$$V(\bar{Y}) = V\left(\frac{1}{n} \sum_{i=1}^n Y_i\right) = \frac{V(Y_i)}{n} = \frac{3}{80n}$$

$$E(Y_i^2) = \int_0^1 3y^4 dy = \frac{3y^5}{5} \Big|_0^1 = \frac{3}{5}$$

$$V(Y_i) = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}$$

$$\lim_{n \rightarrow \infty} V(\bar{Y}) = 0 \Rightarrow \text{consistent so } \bar{Y} \xrightarrow{P} \frac{3}{4}$$

Question 9.36

Question: Suppose that Y has a binomial distribution based on n trials and success probability p . Then $\hat{p}_n = Y/n$ is an unbiased estimator of p . Use Theorem 9.3 to prove that the distribution of

$$Y \sim B(n, p) \quad \hat{p}_n = \frac{Y}{n} \text{ is unbiased estimator for } p.$$

$$\frac{\hat{p}_n - p}{\sqrt{\hat{p}_n \hat{q}_n / n}}$$

converges to a standard normal distribution. [Hint: Write Y as we did in Section

Proof: Define x_i as the i th Bernoulli trial, so

$$Y = \sum_{i=1}^n x_i, \text{ where } x_i = \begin{cases} 1, & \text{if the } i\text{th trial is a success} \\ 0, & \text{otherwise} \end{cases}$$

7.5.]

$$\Rightarrow \hat{p}_n = \frac{Y}{n} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}, \quad E(x_i) = p, \quad V(x_i) = pq$$

$$\text{Define } u_n = \frac{\hat{p}_n - p}{\sqrt{pqn/n}} = \frac{\hat{p}_n - p}{\sqrt{pq/n}} \xrightarrow{d} N(0, 1) \text{ by C.L.T}$$

$$\text{We want } \frac{\hat{p}_n - p}{\sqrt{\hat{p}_n \hat{q}_n / n}} \text{ to be } \frac{u_n}{w_n} \text{ so } w_n = \frac{\sqrt{\hat{p}_n \hat{q}_n / n}}{\sqrt{pq/n}} = \sqrt{\frac{\hat{p}_n \hat{q}_n}{pq}}$$

$$\text{(clearly } \hat{p}_n \xrightarrow{p} p \text{ and } \hat{q}_n \xrightarrow{p} q \Rightarrow \hat{p}_n \hat{q}_n \xrightarrow{p} pq \text{ so clearly } \frac{u_n}{w_n} = \frac{\hat{p}_n - p}{\sqrt{\hat{p}_n \hat{q}_n / n}} \xrightarrow{d} N(0, 1) \text{ by Theorem. } \square$$

$$\text{and } \frac{\hat{p}_n \hat{q}_n}{pq} \xrightarrow{p} 1 \Rightarrow w_n = \sqrt{\frac{\hat{p}_n \hat{q}_n}{pq}} \xrightarrow{p} 1$$

Slutsky Theorem

Theorem 9.3: Suppose that u_n has a distribution function that converges to a standard normal distribution function as $n \rightarrow \infty$. If w_n converges in probability to 1, then the distribution function of $\frac{u_n}{w_n}$ converges to a standard normal distribution function.

Extra Question

Question: Suppose Y_1, Y_2, \dots, Y_n are independent and identically distributed random variables and $E[Y_i] = 0$ and $E[Y_i^2] = 1$ for all $i = 1, 2, \dots, n$. Also assume $E[Y_i^4] < \infty$. Prove the following:

(a) Show $\sqrt{n} \bar{Y} \xrightarrow{d} N(0, 1)$ where \bar{Y} is the sample mean.

$$E(Y_i) = 0, \quad E(Y_i^2) = 1$$

$$V(Y_i) = 1 - 0 = 1$$

$$\Rightarrow \frac{\bar{Y} - 0}{\frac{1}{\sqrt{n}}} \xrightarrow{d} N(0, 1) \text{ by C.L.T}$$

$$\Rightarrow \sqrt{n} \bar{Y} \xrightarrow{d} N(0, 1) \text{ as needed!}$$

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(b) Show $\frac{1}{n} \sum_{i=1}^n Y_i^2 \xrightarrow{P} 1$.

Define $x_i = Y_i^2$, x_1, \dots, x_n are iid

$$\text{so } \frac{1}{n} \sum_{i=1}^n Y_i^2 = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$$

$$\text{Then } E(x_i) = E(Y_i^2) = 1$$

so $E(\bar{x}) = 1$ and $V(\bar{x}) \rightarrow 0$ as $n \rightarrow \infty$ as $E(Y_i^4) = E(x_i^2) < \infty$

$$\text{so, } \frac{1}{n} \sum_{i=1}^n Y_i^2 \xrightarrow{P} 1$$

Extra Question

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(c) Show

$$\frac{\sqrt{n} \bar{Y}}{\frac{1}{n} \sum_{i=1}^n Y_i^2} \xrightarrow{d} N(0, 1).$$

since $\sqrt{n} \bar{Y} \xrightarrow{d} N(0, 1)$
and $\frac{1}{n} \sum_{i=1}^n Y_i^2 \xrightarrow{p} 1$

By Slutsky theorem $\frac{\sqrt{n} \bar{Y}}{\frac{1}{n} \sum_{i=1}^n Y_i^2} \xrightarrow{d} N(0, 1)$