ECO227Y5 Tutorial 20

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Question 9.37

Question: Let $X_1, X_2, \ldots, X_n$ denote n independent and identically distributed Bernoulli random variables such that

$$P(X_i = 1) = p \qquad \text{and} \qquad P(X_i = 0) = 1 - p,$$

for each $i = 1, 2, \ldots, n$. Show that $\sum_{i=1}^n X_i$ is sufficient for p by using the factorization criterion given in Theorem 9.4

Theorem 9.4: Let u be a statistic based on the random sample $Y_1, \ldots, Y_n$. Then u is a sufficient statistic for the estimation of a parameter θ iff the likelihood $L(\theta) = L(Y_1, \ldots, Y_n \mid \theta)$ can be factored into two nonnegative functions,

$$L(Y_1, \ldots, Y_n \mid \theta) = g(u, \theta) \cdot h(Y_1, \ldots, Y_n)$$

where $g(u, \theta)$ is a function only of u and θ and $h(Y_1, \ldots, Y_n)$ is not a function of θ.

$X_1, \ldots, X_n \overset{iid}{\sim} \text{Ber}(p)$

$\Rightarrow P(X_i = x_i) = p^{x_i}(1-p)^{1-x_i}$ where $x_i \in \{0, 1\}$

$$L(x_1, \ldots, x_n \mid p) = \prod_{i=1}^n P(X_i = x_i) = p^{x_1}(1-p)^{1-x_1} \cdot p^{x_2}(1-p)^{1-x_2} \cdots p^{x_n}(1-p)^{1-x_n}$$

$$= p^{\sum x_i}(1-p)^{\sum 1 - x_i} = p^{\sum x_i}(1-p)^{n - \sum x_i}$$

Let $T = \sum_{i=1}^n x_i \Rightarrow L(x_1, \ldots, x_n \mid p) = \underbrace{p^T (1-p)^{n-T}}_{g(T, p)} \cdot 1 \leftarrow h(x_1, \ldots, x_n)$

$\Rightarrow T = \sum_{i=1}^n x_i$ is sufficient for p

Question 9.39

Question: Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a Poisson distribution with parameter λ. Show by conditioning that $\sum_{i=1}^{n} Y_i$ is sufficient for λ.

Recall: T is sufficient for θ if the $P(Y_1, \ldots, Y_n \mid T)$ does not depend on θ.

$Y_1, \ldots, Y_n \overset{iid}{\sim} Poi(\lambda)$

$P(Y_i = y_i) = \frac{e^{-\lambda}\lambda^{y_i}}{y_i!}$

$$\Rightarrow L(Y_1, \ldots, Y_n \mid \lambda) = \frac{e^{-\lambda}\lambda^{y_1}}{y_1!} \cdot \frac{e^{-\lambda}\lambda^{y_2}}{y_2!} \cdots \frac{e^{-\lambda}\lambda^{y_n}}{y_n!} = \frac{e^{-n\lambda}\lambda^{\sum y_i}}{\prod_{i=1}^{n} y_i!}$$

$\sum_{i=1}^{n} y_i$ is automatically defined given $Y_1, \ldots, Y_n$

By conditioning, we compute

$$P(Y_1 = y_1, \ldots, Y_n = y_n \mid \sum_{i=1}^{n} y_i = t) = \frac{P(Y_1 = y_1, \ldots, Y_n = y_n)}{P(\sum_{i=1}^{n} y_i = t)} = \frac{\dfrac{e^{-n\lambda}\lambda^{\sum y_i = t}}{\prod_{i=1}^{n} y_i!}}{\dfrac{e^{-n\lambda}(n\lambda)^t}{t!}} = \frac{t!}{\prod_{i=1}^{n} y_i!} \cdot \frac{1}{n^t}$$

$(n\lambda)^t = n^t\lambda^t$

Key fact: Adding n independent poisson with parameter λ gives a poisson with $n\lambda$.

so $\sum_{i=1}^{n} y_i \sim Poi(n\lambda)$

what we got does not depend on λ so $\sum_{i=1}^{n} y_i$ is sufficient for λ.

Question 9.42

Question: If $Y_1, Y_2, \ldots, Y_n$ denote a random sample from a geometric distribution with parameter p, show that $\bar{Y}$ is sufficient for p.

$Y_1, \ldots, Y_n \overset{iid}{\sim} Geo(p)$

Recall: $P(Y=y) = p(1-p)^{y-1}, \quad y \in \{1, 2, \ldots\}$

$L(y_1, \ldots, y_n \mid p) = \prod_{i=1}^{n} p(Y_i = y_i) = \prod_{i=1}^{n} p(1-p)^{y_i - 1} = p^n (1-p)^{\sum y_i - 1} = p^n (1-p)^{\sum y_i - n}$

let $h(y_1, \ldots, y_n) = 1$, clearly $L(y_1, \ldots, y_n \mid p) \cdot h(y_1, \ldots, y_n) = p^n (1-p)^{\sum y_i - n} \cdot 1 = p^n (1-p)^{n\bar{y} - n} \cdot 1$

satisfies the factorization criterion, and $\bar{y}$ is sufficient for p.

Question 9.44

Question: Let $Y_1, Y_2, \ldots, Y_n$ denote independent and identically distributed random variables from a Pareto distribution with parameters α and β. Then, by the result in Exercise 6.18, if $\alpha, \beta > 0$,

$$f(y \mid \alpha, \beta) = \begin{cases} \alpha\beta^\alpha y^{-(\alpha+1)}, & y \geq \beta, \\ 0, & \text{elsewhere.} \end{cases}$$

If β is known, show that $\sum_{i=1}^n \ln Y_i$ is sufficient for α.

let $\beta = \beta$ known.

$$L(y_1, \ldots, y_n \mid \alpha) = \prod_{i=1}^{n} \alpha \beta^\alpha y_i^{-(\alpha+1)} = \alpha \beta^\alpha y_1^{-(\alpha+1)} \cdot \alpha \beta^\alpha y_2^{-(\alpha+1)} \cdots \alpha \beta^\alpha y_n^{-(\alpha+1)}$$

$$= \alpha^n \beta^{n\alpha} \left(\prod_{i=1}^n y_i\right)^{-(\alpha+1)} = \alpha^n \beta^{n\alpha} \left(\prod_{i=1}^n y_i\right)^{-\alpha} \cdot \left(\prod_{i=1}^n y_i\right)^{-1} = \underbrace{\alpha^n \beta^{n\alpha} e^{-\alpha \sum_{i=1}^n \ln y_i}}_{g(T, \alpha)} \cdot \underbrace{\frac{1}{\prod_{i=1}^n y_i}}_{h(y_1, \ldots, y_n)}$$

Ex. $(x \cdot y)^{-\alpha}$

$= x^{-\alpha} \cdot y^{-\alpha}$

$= e^{\ln(x^{-\alpha})} \cdot e^{\ln(y^{-\alpha})}$

$= e^{-\alpha \ln(x)} \cdot e^{-\alpha \ln(y)}$

$= e^{-\alpha(\ln(x) + \ln(y))}$

so $\sum_{i=1}^n \ln y_i$ is sufficient for α

Question 9.60

Question: Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the probability density function

$$f(y \mid \theta) = \begin{cases} \theta y^{\theta - 1}, & 0 < y < 1, \ \theta > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Show that this density function is in the one-parameter exponential family and that

$$\sum_{i=1}^{n} -\ln(Y_i)$$

is sufficient for θ. (See Exercise 9.45.)

Recall: $f(y \mid \theta) \in$ one parameter exponential family if $f(y \mid \theta) = h(y)\, c(\theta)\, e^{w(\theta) t(y)}$

Gamma: $f(y) = \dfrac{y^{\alpha - 1} e^{-y/\beta}}{\Gamma(\alpha)\beta^{\alpha}}, \ y > 0$

if $\alpha = 1$, $\Rightarrow f(y) = \dfrac{e^{-y/\beta}}{\beta}$

$$f(y \mid \theta) = \theta y^{\theta - 1} = \theta e^{\ln(y)(\theta - 1)} = \underbrace{y^{-1}}_{h(y)} \underbrace{\theta}_{c(\theta)} e^{\ln(y)\theta}$$

$$L(y_1, \ldots, y_n \mid \theta) = \prod_{i=1}^{n} \theta y_i^{\theta - 1} = \theta y_1^{\theta - 1} \cdot \theta y_2^{\theta - 1} \cdots \theta y_n^{\theta - 1} = \theta^n \left(\prod_{i=1}^{n} y_i\right)^{\theta - 1} = \theta^n \left(\prod_{i=1}^{n} y_i\right)^{\theta} \cdot \frac{1}{\prod_{i=1}^{n} y_i}$$

$$= \underbrace{\theta^n e^{-\theta \sum_{i=1}^{n} -\ln(y_i)}}_{g(t, \theta)} \cdot \underbrace{\frac{1}{\prod_{i=1}^{n} y_i}}_{h(y_1, \ldots, y_n)}$$

so $\sum_{i=1}^{n} -\ln(y_i)$ is sufficient for θ

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Question 9.64

Question: Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from a normal distribution with mean μ and variance 1.

(a) Show that the MVUE of μ^2 is

$$\hat{\mu}^2 = \bar{Y}^2 - \frac{1}{n}.$$

(b) Derive the variance of $\widehat{\mu^2}$.

a.) Theorem: (Lehmann-Scheffé Theorem)

let $X_1, \ldots, X_n$ be a random sample from a distribution with parameter θ. If:

1. $T(X_1, \ldots, X_n)$ is a sufficient statistic for θ, and
2. $h(T(X_1, \ldots, X_n))$ is an unbiased estimator for θ. ($E(h(T(X_1,\ldots,X_n))) = \theta$) for all θ.

Then $h(T(X_1,\ldots,X_n))$ is the unique MVUE of θ

since σ^2 is known, $\bar{Y}$ is sufficient for μ. $E(\hat{\mu}^2) = E(\bar{Y}^2 - \frac{1}{n}) = E(\bar{Y}^2) - \frac{1}{n} = V(\bar{Y}) + E(\bar{Y})^2 - \frac{1}{n} = \frac{1}{n} + \mu^2 - \frac{1}{n} = \mu^2$

so $\hat{\mu}^2$ is MVUE

b.) $V(\hat{\mu}^2) = V(\bar{Y}^2 - \frac{1}{n}) = V(\bar{Y}^2) = E(\bar{Y}^4) - E(\bar{Y}^2)^2 = \frac{3}{n^2} + \frac{6\mu^2}{n^2} + \mu^4 - (\frac{1}{n} + \mu^2)^2 = \frac{2 + 4n\mu^2}{n^2}$

For $Y \sim N(\mu, \sigma^2)$, $E(Y^4) = 3\sigma^4 + 6\mu^2\sigma^2 + \mu^4$, so with $\bar{Y} \sim N(\mu, \frac{1}{n})$, $E(\bar{Y}^4) = 3(\frac{1}{n})^2 + 6\mu^2(\frac{1}{n})^2 + \mu^4$

Try it with $\bar{Y} = \mu + \frac{Z}{\sqrt{n}}$, $Z \sim N(0,1)$ then expand $\bar{Y}^4$

Question 9.69

Question: Let $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the probability density function

$$f(y \mid \theta) = \begin{cases} (\theta+1)y^{\theta}, & 0 < y < 1,\ \theta > -1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find an estimator for θ by the method of moments. Show that the estimator is consistent. Is the estimator a function of the sufficient statistic

Big Idea: Count how many unknown parameters you have. If you have k unknowns, you need k equations, get the k equations from setting the first k population moments equal to their sample counterparts, then solve for the parameter.

$$\sum_{i=1}^{n} \ln(Y_i)$$

that we can obtain from the factorization criterion? What implications does this have?

$$E(Y) = (\theta+1)\int_0^1 y^{\theta+1}\,dy = (\theta+1)\frac{y^{\theta+2}}{\theta+2}\Big|_0^1 = \frac{\theta+1}{\theta+2}$$

$$\bar{Y} = E(Y) = \frac{\theta+1}{\theta+2} \Rightarrow \bar{Y}\theta + 2\bar{Y} = \theta + 1 \Rightarrow \bar{Y}\theta - \theta = 1 - 2\bar{Y} \Rightarrow \theta(\bar{Y}-1) = 1-2\bar{Y} \Rightarrow \hat{\theta}_{mom} = \frac{1-2\bar{Y}}{\bar{Y}-1}$$

$$E(\hat{\theta}_{mom}) = E\left(\frac{1-2\bar{Y}}{\bar{Y}-1}\right) = \frac{1-2\left(\frac{\theta+1}{\theta+2}\right)}{\frac{\theta+1}{\theta+2}-1} = \theta \quad \text{so } \hat{\theta}_{mom} \xrightarrow{P} \theta.$$

From factorization, $\sum_{i=1}^{n} Y_i$ is sufficient, by Rao-Blackwell, a better unbiased estimator exists

Question 9.72

Question: If $Y_1, Y_2, \ldots, Y_n$ denote a random sample from the normal distribution with mean μ and variance σ^2, find the method-of-moments estimators of μ and σ^2.

2 unknown parameters, so two equations needed

$E(Y) = \mu \Rightarrow \hat{\mu} = \bar{Y}$

$E(Y^2) = \sigma^2 + \mu^2 = \frac{1}{n}\sum Y_i^2 \Rightarrow \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \mu^2$ where $\hat{\mu} = \bar{Y}$ — remark: This is a biased estimator

$\Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} Y_i^2 - \bar{Y}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$

Question 9.75

Question: Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from the probability density function

Beta: $f(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}$

$$f(y \mid \theta) = \begin{cases} \dfrac{\Gamma(2\theta)}{[\Gamma(\theta)]^2}\, y^{\theta-1}(1-y)^{\theta-1}, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the method-of-moments estimator for θ.

$Y_i \overset{iid}{\sim} \text{Beta}(\theta, \theta)$

$E(Y) = \frac{\theta}{2\theta} = \frac{1}{2}$ no information, find next moment

$E(Y^2) = V(Y) + E(Y)^2 = \frac{\theta^2}{4\theta^2(2\theta+1)} + \frac{1}{4} = \frac{1}{n}\sum_{i=1}^{n} y_i^2 = m_2$

$\Rightarrow m_2 - \frac{1}{4} = \frac{1}{4(2\theta+1)} \Rightarrow \hat{\theta}_{mom} = \frac{1-2m_2}{4m_2-1}$

Question 9.77

Question: Let $Y_1, Y_2, \ldots, Y_n$ denote independent and identically distributed uniform random variables on the interval $(0, 3\theta)$. Derive the method-of-moments estimator for θ.

$Y_1, \ldots, Y_n \overset{iid}{\sim} uni(0, 3\theta)$

$E(Y) = \frac{3\theta}{2} = \bar{Y} \Rightarrow \hat{\theta}_{mom} = \frac{2\bar{Y}}{3}$