ECO227Y5 Tutorial 21

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Question: Suppose that $Y_1, Y_2, \ldots, Y_n$ constitute a random sample from a uniform distribution with probability density function

$$f(y \mid \theta) = \begin{cases} \dfrac{1}{2\theta+1}, & 0 \le y \le 2\theta+1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Obtain the MLE of θ.

- strictly decreasing in θ

$$L(y_1, \ldots, y_n \mid \theta) = \prod_{i=1}^{n} \frac{1}{2\theta+1} = \frac{1}{(2\theta+1)^n} \, I(y_{(n)} \le 2\theta+1) \, I(y_{(1)} \ge 0)$$

$$\ln(L(y_i \mid \theta)) = \ln\left(\frac{1}{(2\theta+1)^n}\right) = \underbrace{\ln(1)}_{=0} - n\ln(2\theta+1)$$

$$\Rightarrow \ln(L(y_i \mid \theta)) = -n\ln(2\theta+1)$$

$$\frac{\partial \ln(L(y_i \mid \theta))}{\partial \theta} = \frac{-2n}{2\theta+1} = 0 \quad \text{does not work!!}$$

Try: want $\hat{\theta}$ as small as possible.
but it can't be too small.
The support must cover all the data
so $Y_{(n)} \le 2\theta + 1 \Rightarrow \hat{\theta}_{MLE} = \frac{Y_{(n)} - 1}{2}$

Question 9.89

Question: It is known that the probability p of tossing heads on an unbalanced coin is either $\frac{1}{4}$ or $\frac{3}{4}$. The coin is tossed twice and a value for Y, the number of heads, is observed. For each possible value of Y, which of the two values for p ($\frac{1}{4}$ or $\frac{3}{4}$) maximizes the probability that $Y = y$? Depending on the value of y actually observed, what is the MLE of p?

$Y \sim Bin(2,p)$

$L(Y|p) = \binom{2}{p} p^Y (1-p)^{2-Y}, \quad Y \in \{0,1,2\}$

$y=0: \binom{2}{0}(1-p)^2 = (1-p)^2 = 1-2p+p^2$

$y=1: \binom{2}{1} p(1-p) = 2p-2p^2$

$y=2: \binom{2}{2} p^2 = p^2$

$y=0: L(0|p) = 1-2p+p^2$

$p=\frac{1}{4}: L(0|\frac{1}{4}) = 1-\frac{2}{4}+\frac{1}{16} = \frac{9}{16}$

$p=\frac{3}{4}: L(0|\frac{3}{4}) = 1-\frac{6}{4}+\frac{9}{16} = \frac{1}{16}$

$\hat{p}_{MLE} = \frac{1}{4}$

$y=1: L(1|p) = 2p-2p^2$

$p=\frac{1}{4}: L(1|\frac{1}{4}) = \frac{2}{4}-\frac{2}{16} = \frac{3}{8}$

$p=\frac{3}{4}: L(1|\frac{3}{4}) = \frac{6}{4}-\frac{18}{16} = \frac{3}{8}$

both equally good!

$\hat{p}_{MLE} = \frac{1}{4}$ or $\frac{3}{4}$

$y=2: L(2|p) = p^2$

$p=\frac{1}{4}: L(2|\frac{1}{4}) = \frac{1}{16}$

$p=\frac{3}{4}: L(2|\frac{3}{4}) = \frac{9}{16}$

$\hat{p}_{MLE} = \frac{3}{4}$

Question 9.97

Question: The geometric probability mass function is given by

$$p(y \mid p) = p(1-p)^{y-1}, \qquad y = 1, 2, 3, \ldots$$

A random sample of size n is taken from a population with a geometric distribution.

(a) Find the method-of-moments estimator for p.

(b) Find the MLE for p.

a.) $E(Y) = \frac{1}{p} = \bar{Y} \Rightarrow \hat{p} = \frac{1}{\bar{Y}}$

b.) $L(Y_1, \ldots, Y_n \mid p) = \prod_{i=1}^{n} p(1-p)^{Y_i - 1}$ take ln

$\Rightarrow \ln(L(Y_i \mid p)) = \sum_{i=1}^{n} \underbrace{\ln(p)}_{= n\ln(p)} + (Y_i - 1)\ln(1-p)$

$\frac{\partial \ln(L(Y_i \mid p))}{\partial p} = \frac{n}{p} - \frac{\sum_{i=1}^{n}(Y_i - 1)}{1-p} = 0 \Rightarrow \frac{n - np - p\sum_{i=1}^{n}(Y_i - 1)}{p - p^2} = 0 \Rightarrow n - np - p\sum_{i=1}^{n}(Y_i - 1) = 0$

$\Rightarrow n = p\left(n + \sum_{i=1}^{n}(Y_i - 1)\right)$

$\Rightarrow p = \frac{n}{n + \sum_{i=1}^{n}(Y_i - 1)} = \frac{n}{n + \sum_{i=1}^{n} Y_i - n} = \frac{n}{\sum_{i=1}^{n} Y_i} = \frac{1}{\bar{Y}}$

Question 10.6

Question: We are interested in testing whether or not a coin is balanced based on the number of heads Y on 36 tosses of the coin ($H_0 : p = 0.5$ versus $H_a : p \neq 0.5$). If we use the rejection region $|y - 18| \geq 4$, what is

(a) the value of α?

(b) the value of β if $p = 0.7$?

Type I error

$\alpha = P(\text{Reject } H_0 \mid H_0 \text{ true})$

Can use Z.

$Y \sim Bin(36, p)$

$\mu = np$

$\sigma^2 = np(1-p)$

a.) $|Y-18| \geq 4 \Rightarrow Y - 18 \geq 4$ or $Y - 18 \leq -4$

$\Rightarrow Y \geq 22$ or $Y \leq 14$

$P(Y \geq 22 \text{ or } Y \leq 14 \mid p = 0.5) = \alpha$

$P(Y \geq 22 \mid p = 0.5) + P(Y \leq 14 \mid p = 0.5) = \alpha$

μ from $p = 0.5$

$= P\left(Z \geq \frac{22-18}{3}\right) + P\left(Z \leq \frac{14-18}{3}\right) = P\left(Z \geq \frac{4}{3}\right) + P\left(Z \leq \frac{-4}{3}\right) = 2P\left(Z \geq \frac{4}{3}\right) = \alpha \Rightarrow \alpha \simeq 0.1876$

Type II error

b.) $\beta = P(\text{Fail to reject } H_0 \mid H_0 \text{ false})$

- so H_0 false

fail to reject so

$|Y-18| < 4$

$\Rightarrow -4 < Y - 18 < 4 \Rightarrow 14 < Y < 22 \Rightarrow 15 \leq Y \leq 21$

$\Rightarrow P(15 \leq Y \leq 21 \mid p = 0.7) = \beta$

$\Rightarrow P\left(Z \geq \frac{15 - 0.7 \cdot 36}{\sqrt{36 \cdot 0.7 \cdot 0.3}}\right) + P\left(Z \leq \frac{21 - 0.7 \cdot 36}{\sqrt{36 \cdot 0.7 \cdot 0.3}}\right) = \beta$

$\Rightarrow P(Z \geq -3.71) + P(Z \leq -1.53) = \beta \Rightarrow \beta = 0.0633$

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Question 10.7

Question: *True or False.* Refer to Exercise 10.6.

(a) The level of the test computed in Exercise 10.6(a) is the probability that H_0 is true.

(b) The value of β computed in Exercise 10.6(b) is the probability that H_a is true.

(c) In Exercise 10.6(b), β was computed assuming that the null hypothesis was false.

(d) If β was computed when $p = 0.55$, the value would be larger than the value of β obtained in Exercise 10.6(b).

a.) False $\alpha = P($Reject $H_0 \mid H_0$ is true$)$

b.) False $\beta = P($fail to reject $H_0 \mid H_0$ is false$)$

c.) True.

d.) True as it is closer to out H_0

Question 10.7

Question: *True or False.* Refer to Exercise 10.6.

(e) The probability that the test mistakenly rejects H_0 is β.

(f) Suppose that RR was changed to $|y - 18| \geq 2$. – bigger RR

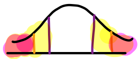
  (i) This RR would lead to rejecting the null hypothesis more often than the RR used in Exercise 10.6.

  (ii) If α was computed using this new RR, the value would be larger than the value obtained in Exercise 10.6(a).

  (iii) If β was computed when $p = 0.7$ and using this new RR, the value would be larger than the value obtained in Exercise 10.6(b).

e.) False, it is α.

f.) i.) True, it is bigger now



ii.) True

iii.) False, it decreases as β is related to not rejecting.

Question 10.17

Question: A survey published in the *American Journal of Sports Medicine* reported the number of meters (m) per week swum by two groups of swimmers — those who competed exclusively in breaststroke and those who competed in the individual medley (which includes breaststroke). The summary statistics are given below.

	μ_1	μ_2
Specialty	**Exclusively Breaststroke**	**Individual Medley**
Sample size	130	80
Sample mean (m)	9017	5853
Sample SD (m)	7162	1961
Population mean	μ_1	μ_2

Is there sufficient evidence that the average meters per week practicing breaststroke is greater for exclusive breaststrokers than for individual medley swimmers?

(a) State the null and alternative hypotheses.

$H_0: \mu_1 - \mu_2 = 0$

$H_1: \mu_1 - \mu_2 > 0$

Question 10.17

Question:

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Is there sufficient evidence that the average meters per week practicing breaststroke is greater for exclusive breaststrokers than for individual medley swimmers?

(b) What is the appropriate rejection region for an $\alpha = 0.01$ level test?

(c) Calculate the observed value of the appropriate test statistic.

b.)

$\alpha = 0.01$

$n_1 = 130,\ n_2 = 80$

$P(Z \geq a) = 0.01 \Rightarrow a \approx 2.33$

c.) $z = \dfrac{\bar{y}_1 - \bar{y}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \dfrac{9017 - 5853}{\sqrt{\frac{7162^2}{130} + \frac{1961^2}{80}}} = 4.76$

Question:

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Is there sufficient evidence that the average meters per week practicing breaststroke is greater for exclusive breaststrokers than for individual medley swimmers?

(d) What is your conclusion?

(e) What is a practical reason for the conclusion in part (d)?

d.) Since Test stat yielded $z = 4.76$ and it is bigger than 2.33 we reject H_0 and accept H_a.

e.) Part D tells us we are 99% confident that the average number of meters per week spent practicing breaststroke is higher for exclusive breaststroke swimmers. As swimmers who compete in individual medley have to practice other styles.

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Question 10.19

Question: The output voltage for an electric circuit is specified to be 130. A sample of 40 independent readings on the voltage for this circuit gave a sample mean of 128.6 and standard deviation of 2.1. Test the hypothesis that the average output voltage is 130 against the alternative that it is less than 130. Use a test with level $\alpha = 0.05$.

$\bar{y} = 128.6$, $n = 40$ use z

sd = 2.1

$H_0: \mu = 130$

$H_a: \mu < 130$

test stat: $z = \dfrac{128.6 - 130}{\sqrt{\frac{2.1^2}{40}}} = -4.22$

$P(z \le \alpha) = 0.05 \Rightarrow \alpha = -1.645$

clearly $z = -4.22 < -1.645$ so reject H_0

and accept H_a.