ECO227Y5 Tutorial 4

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Question 3.66

Question: Suppose that Y is a random variable with a geometric distribution. Show that
A: $\sum_y p(y) = \sum_{y=1}^{\infty} q^{y-1}p = 1$.
B: $\frac{p(y)}{p(y-1)} = q$, for $y = 2, 3, \ldots$ This ratio is less than 1, implying that the geometric probabilities are monotonically decreasing as a function of y. If Y has a geometric distribution, what value of Y is the most likely (has the highest probability)?

, geometric series, clearly $1-p \in (-1, 1)$

Note: $\sum_{i=1}^{\infty} a(r)^{i-1}$ converges iff $r \in (-1,1)$

$\sum_{i=1}^{\infty} a(r)^{i-1} = \frac{a}{1-r}$

a.) $\sum_{y=1}^{\infty} q^{y-1}p = \sum_{y=1}^{\infty} p(1-p)^{y-1} = \frac{p}{1-(1-p)} = \frac{p}{p} = 1$ ■

b.) $\frac{p(y)}{p(y-1)} = \frac{q^{y-1}p}{q^{y-1-1}p} = \frac{q^{y-1}p}{q^{y-2}p} = q$ if $y \in \{2, 3, \ldots\}$

Since this is decreasing, the value of Y that is most likely is $Y = 1$ as $P(Y=1) = p$

Question 3.71

Question: Let Y denote a geometric random variable with probability of success p.
A: Show that for a positive integer a, $P(Y > a) = q^a$.
B: Show that for positive integers a and b,
$P(Y > a + b \mid Y > a) = q^b = P(Y > b)$. This result implies that, for example, $P(Y > 7 \mid Y > 2) = P(Y > 5)$. Why do you think this property is called the memoryless property of the geometric distribution?

Note: Partial sum of geometric series $\sum_{i=1}^{n} a(r)^{i-1} = S_n = \frac{a(1-r^n)}{1-r}$, $r \neq 1$

a.) Let $a > 0$, $a \in \mathbb{Z}$. $P(Y>a) = 1 - P(Y \le a) = 1 - \sum_{y=1}^{a} pq^{y-1}$

$$\Rightarrow 1 - \sum_{y=1}^{a} pq^{y-1} = 1 - \left(\frac{p(1-q^a)}{1-q} \right) = 1 - (1-q^a) = q^a \blacksquare$$

($p = 1-q$)

b.) Consider $P(Y>a+b \mid Y>a) = \frac{P(Y>a+b \text{ and } Y>a)}{P(Y>a)} = \frac{P(Y>a+b)}{P(Y>a)} = \frac{q^{a+b}}{q^a} = q^b = P(Y>b)$

This is called the memoryless property as the conditional distribution does not depend on past failures; only the number of future trials matter.

Question 3.71

Question: Let Y denote a geometric random variable with probability of success p.

C: In the development of the distribution of the geometric random variable, we assumed that the experiment consisted of conducting identical and independent trials until the first success was observed. In light of these assumptions, why is the result in part (b) "obvious"?

- independent and identical

C. This is obvious as since iid, after a failures, the process restarts fresh with success probability p.

Question 3.86

Question: Consider an extension of the situation discussed in Example 3.13. If we observe y_0 as the value for a geometric random variable Y, show that $P(Y = y_0)$ is maximized when $p = \frac{1}{y_0}$. Again, we are determining (in general this time) the value of p that maximizes the probability of the value of Y that we actually observed.

$$P(Y = y_0) = q^{y_0 - 1} p = (1-p)^{y_0-1} p$$

$$\frac{dP(Y=y_0)}{dp} = -(y_0-1)(1-p)^{y_0-2} p + (1-p)^{y_0-1} = 0 \quad \text{First order conditions}$$

$$\Rightarrow (1-p)^{y_0-2}\left[-(y_0-1)p + (1-p)\right] = 0$$

$$\Rightarrow (1-p)^{y_0-2} = 0 \quad \text{doesn't tell us anything except } p=1$$

$$\Rightarrow -(y_0-1)p + (1-p) = 0 \Rightarrow 1-p = (y_0-1)p \Rightarrow \frac{1}{p} - 1 = y_0 - 1 \Rightarrow \frac{1}{p} = y_0 \Rightarrow p = \frac{1}{y_0} \quad \blacksquare$$

Question 3.122

Question: Customers arrive at a checkout counter in a department store according to a Poisson distribution at an average of seven per hour. During a given hour, what are the probabilities that

A: no more than three customers arrive?

B: at least two customers arrive?

C: exactly five customers arrive?

$Y \sim Poi(\lambda)$

Recall: $P(Y=y) = \frac{\lambda^y e^{-\lambda}}{y!}$

$Y \sim Poi(7)$, $n = 1$ hour

a.) $P(Y \le 3) = P(Y=0) + P(Y=1) + P(Y=2) + P(Y=3) = \frac{7^0 e^{-7}}{0!} + \frac{7^1 e^{-7}}{1!} + \frac{7^2 e^{-7}}{2!} + \frac{7^3 e^{-7}}{3!} = 0.0818$

b.) $P(Y \ge 2) = 1 - P(Y < 2) = 1 - P(Y \le 1) = 1 - (P(Y=0) + P(Y=1)) = 1 - \left(\frac{7^0 e^{-7}}{0!} + \frac{7^1 e^{-7}}{1!}\right)$

$= 0.993$

c.) $P(Y=5) = \frac{7^5 e^{-7}}{5!} = 0.128$

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Question: Let Y have a Poisson distribution with mean λ. Find $E[Y(Y-1)]$ and then use this to show that $V(Y) = \lambda$.

let $Y \sim poi(\lambda)$. $E(Y(Y-1)) = \sum_{y=0}^{\infty} y(y-1)P(Y=y) = \sum_{y=0}^{\infty} y(y-1)\frac{\lambda^y e^{-\lambda}}{y!} = \sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{(y-2)!} = e^{-\lambda}\sum_{y=0}^{\infty}\frac{\lambda^y}{(y-2)!}$

$y! = y\cdot(y-1)\cdots 1$

Let us re-index the sum: $e^{-\lambda}\sum_{y=0}^{\infty}\frac{\lambda^y}{(y-2)!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k+2}}{k!} = e^{-\lambda}\lambda^2\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$

Note: The maclaurin series for $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$= e^{-\lambda}\lambda^2 \cdot e^{\lambda} = \lambda^2$

Recall: $V(Y) = E(Y^2) - E(Y)^2$, $E(Y(Y-1)) = E(Y^2 - Y) = E(Y^2) - \underbrace{E(Y)}_{=\lambda} = \lambda^2 \Rightarrow E(Y^2) = \lambda^2 + \lambda$

$V(Y) = \lambda^2 + \lambda - \lambda^2 = \lambda$ as needed.

Question 3.143

Question: Refer to Exercise 3.142 (c). If the number of phone calls to the fire department, Y , in a day has a Poisson distribution with mean 5.3, what is the most likely number of phone calls to the fire department on any day?

mode

$Y \sim Poi(5.3)$. Recall that Y is discrete so it can only take on integer values.

Consider the ratio between $\frac{p(Y=y+1)}{p(Y=y)} = \frac{\lambda^{y+1}/(y+1)!}{\lambda^{y}/y!} = \frac{\lambda^{y+1}\,y!}{\lambda^{y}\,(y+1)!} = \frac{\lambda}{y+1}$, if this ratio >1, we can increase probability by increasing y

if this ratio <1, probability decreases as we increase Y

want $\frac{\lambda}{y+1} \simeq 1 \Rightarrow \lambda \simeq y+1 \Rightarrow y \simeq \lambda - 1$

floor function

For non-integer λ, the mode $= \lfloor \lambda \rfloor$. For integer λ, both λ and $\lambda-1$ are modes

So $\lambda = 5.3 \Rightarrow$ mode $= \lfloor 5.3 \rfloor = 5$

Why Both λ and $\lambda - 1$?

$$p(Y=\lambda) = \frac{\lambda^{\lambda} e^{-\lambda}}{\lambda!} = \frac{\cancel{\lambda} \cdot \lambda^{\lambda-1} e^{-\lambda}}{\cancel{\lambda}(\lambda-1)!} = \frac{\lambda^{\lambda-1} e^{-\lambda}}{(\lambda-1)!} = p(Y=\lambda-1)$$

$$p(Y=\lambda-1) = \frac{\lambda^{\lambda-1} e^{-\lambda}}{(\lambda-1)!} = p(Y=\lambda)$$