ECO227Y5 Tutorial 5

William Hsu

Department of Economics
University of Toronto Mississauga

October 20th, 2025

Question 3.156

Question: Suppose that Y is a random variable with moment-generating function $m(t)$.
A: What is $m(0)$?
B: If $W = 3Y$, show that the moment-generating function of W is $m(3t)$.
C: If $X = Y - 2$, show that the moment-generating function of X is $e^{-2t}m(t)$.

a.) $m(t) = E(e^{tY}) = \sum_{i=0}^{n} e^{tY} \cdot p(Y=y)$

$m(0) = E(e^{0}) = E(1) = 1$

$W=3Y$

b.) $m_Y(t) = E(e^{tY}) \Rightarrow m_W(t) = E(e^{tW}) = E(e^{t(3Y)}) = E(e^{(3t)Y}) = m_Y(3t)$

c.) $m_Y(t) = E(e^{tY}) \Rightarrow m_X(t) = E(e^{tX}) = E(e^{t(Y-2)}) = E(e^{tY-2t}) = E(e^{tY} \cdot e^{-2t})$

notice that e^{-2t} is a constant when we take the expectation as Y is the random variable.

$\Rightarrow e^{-2t} E(e^{tY}) = e^{-2t} m_Y(t)$

Question 3.158

Question: If Y is a random variable with moment-generating function $m(t)$ and if W is given by

$$W = aY + b,$$

show that the moment-generating function of W is $e^{tb}m(at)$.

$$m_Y(t) = E(e^{tY})$$

$$m_W(t) = E[e^{tW}] = E(e^{t(aY+b)}] = E(e^{atY+tb}) = E(e^{atY} \cdot e^{tb})$$

Again e^{tb} is a constant

$$\Rightarrow e^{tb} E(e^{(at)Y}) = e^{tb} m_Y(at).$$

Question 4.5

Question: Suppose that Y is a random variable that takes on only integer values $1, 2, \ldots$ and has distribution function $F(y)$. Show that the probability function $p(y) = P(Y = y)$ is given by

$$p(y) = \begin{cases} F(1), & y = 1, \\ F(y) - F(y-1), & y = 2, 3, \ldots \end{cases}$$

$Y \in \{1, 2, \ldots\}$

$P(Y=1) = P(Y \leq 1) = F(1)$ as 1 is the lowest value that Y can take on.

as there are no values between $y-1$ and y

$P(Y = y \geq 2) = P(Y \leq y) - P(Y < y) = P(Y \leq y) - P(Y \leq y-1) = F(y) - F(y-1)$

Question 4.8

Question: Suppose that Y has density function

Valid PDF: $\int_{-\infty}^{\infty} f(y)\,dy = 1$, $f(y) \geq 0$ for all y

$$f(y) = \begin{cases} k\,y(1-y), & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of k that makes $f(y)$ a probability density function.

(b) Find $P(0.4 \leq Y \leq 1)$.

(c) Find $P(0.4 \leq Y < 1)$.

Notice for any $y \notin [0,1]$, $f(y) = 0$ so,

a.) $\int_{-\infty}^{\infty} ky(1-y)\,dy = \int_0^1 ky(1-y)\,dy = k\int_0^1 y - y^2\,dy$

$= k\left[\frac{y^2}{2} - \frac{y^3}{3}\right]\Big|_0^1 = \frac{k}{2} - \frac{k}{3} = 1$ ← valid PDF

$\Rightarrow \frac{k}{6} = 1 \Rightarrow k = 6$, $f(y) = \begin{cases} 6y(1-y), & y \in [0,1] \\ 0, & \text{elsewhere} \end{cases}$

b.) $P(0.4 \leq Y \leq 1) = \int_{0.4}^1 6y(1-y)\,dy$

$= 6\int_{0.4}^1 y - y^2\,dy = 6\left[\frac{y^2}{2} - \frac{y^3}{3}\right]\Big|_{0.4}^1$

$= 6\left[\frac{27}{250}\right] = 0.648$

c.) Recall that for any continuous Y,

$\int_a^a f(y)\,dy = F(a) - F(a) = 0$

so $P(0.4 \leq Y < 1) = P(0.4 \leq Y \leq 1) = 0.648$

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Question 4.8

Question: Suppose that Y has density function

$$f(y) = \begin{cases} k\,y(1-y), & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(d) Find $P(Y \le 0.4 \mid Y \le 0.8)$.

(e) Find $P(Y < 0.4 \mid Y < 0.8)$.

d.) we find F(y) to take less integrals.

$$F(y) = \int_0^y 6y(1-y)\,dy = 6\int_0^y y - y^2\,dy = 6\left[\frac{y^2}{2} - \frac{y^3}{3}\right]\Bigg|_0^y = 3y^2 - 2y^3 \text{ if } y \in [0,1]$$

$$F(0.4) = 3(0.4)^2 - 2(0.4)^3 = 0.352, \quad F(0.8) = 3(0.8)^2 - 2(0.8)^3 = 0.896$$

$$P(Y \le 0.4 \mid Y \le 0.8) = \frac{P(Y \le 0.4 \wedge Y \le 0.8)}{P(Y \le 0.8)} = \frac{0.352}{0.896} = \frac{11}{28} \approx 0.3929$$

d.) continuous so $P(Y < 0.4 \mid Y < 0.8) = 0.3929$

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Question 4.9

Question: A random variable Y has the following distribution function:

$$F(y) = P(Y \leq y) = \begin{cases} 0, & y < 2, \\ \frac{1}{8}, & 2 \leq y < 2.5, \\ \frac{3}{16}, & 2.5 \leq y < 4, \\ \frac{1}{2}, & 4 \leq y < 5.5, \\ \frac{5}{8}, & 5.5 \leq y < 6, \\ \frac{11}{16}, & 6 \leq y < 7, \\ 1, & y \geq 7. \end{cases}$$

(a) Is Y a continuous or discrete random variable? Why?

(b) What values of Y are assigned positive probabilities?

a.) Discrete, the CDF has jumps.

b.) Notice $F(y)$ is constant at certain y's, and only increases on endpoints of intervals.

So, $y = 2, 2.5, 4, 5.5, 6, 7$ have positive probabilities

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Question 4.9

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(c) Find the probability function for Y.

(d) What is the median, $\phi_{0.5}$, of Y?

c.) $f(y) = \begin{cases} \frac{1}{8} & y=2 \\ \frac{3}{16}-\frac{1}{8}=\frac{1}{16} & y=2.5 \\ \frac{5}{16} & y=4 \\ \frac{1}{8} & y=5.5 \\ \frac{1}{16} & y=6 \\ \frac{5}{16} & y=7 \\ 0, & \text{elsewhere} \end{cases}$

d.) $\phi_{0.5} \Rightarrow F(y) = 0.5$, clearly we see that $F(4) = \frac{1}{2}$.

so $\phi_{0.5} = 4$

Question: Suppose that Y possesses the density function

$$f(y) = \begin{cases} c\,y, & 0 \le y \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of c that makes $f(y)$ a probability density function.

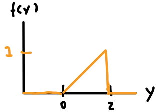
(b) Find $F(y)$.

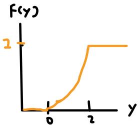
(c) Graph $f(y)$ and $F(y)$.

a.) $\int_{-\infty}^{\infty} cy\,dy = 1 \Rightarrow c\int_0^2 y\,dy = 1 \Rightarrow c\left[\frac{y^2}{2}\right]\Big|_0^2 = 1 \Rightarrow 2c = 1 \Rightarrow c = \frac{1}{2}$

b.) $F(y) = \int_0^y \frac{1}{2}y\,dy = \frac{1}{2}\int_0^y y\,dy = \frac{1}{2}\left(\frac{y^2}{2}\right)\Big|_0^y = \frac{y^2}{4} \Rightarrow F(y) = \begin{cases} 0, & y < 0 \\ \frac{y^2}{4}, & 0 \le y \le 2 \\ 1, & y > 2 \end{cases}$

c.)





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Question 4.11

Question: Suppose that Y possesses the density function

$$f(y) = \begin{cases} c\,y, & 0 \le y \le 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(d) Use $F(y)$ to find $P(1 \le Y \le 2)$.

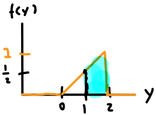
(e) Use $f(y)$ and geometry to find $P(1 \le Y \le 2)$.

d.) $P(1 \le Y \le 2) = P(Y \le 2) - P(Y \le 1) = F(Y=2) - F(Y=1) = 1 - \frac{1}{4} = \frac{3}{4}$

$F(2) = 1$, $F(1) = \frac{1}{4}$

Area of Trapezoid: $A = \frac{(\text{Base} + \text{Top}) \cdot \text{Height}}{2}$

e.)



$P(1 \le Y \le 2) = \frac{(\frac{1}{2} + 1) \cdot 1}{2} = \frac{3}{4}$

Question 4.12

Question: The length of time to failure (in hundreds of hours) for a transistor is a random variable Y with distribution function

$$F(y) = \begin{cases} 0, & y < 0, \\ 1 - e^{-y^2}, & y \geq 0. \end{cases}$$

(a) Show that $F(y)$ has the properties of a distribution function.

(b) Find the 0.30-quantile, $\phi_{0.30}$, of Y.

a.) check limits:

$\lim_{y \to -\infty} F(y) = 0$ clearly. $\lim_{y \to \infty} F(y) = \lim_{y \to \infty} (1 - e^{-y^2}) = 1 - \lim_{y \to \infty} \frac{1}{e^{y^2}} = 1 - 0 = 1$

b.) $\phi_{0.30} \Rightarrow F(y) = 0.30 \Rightarrow 1 - e^{-y^2} = 0.30 \Rightarrow e^{-y^2} = 0.70 \Rightarrow -y^2 = \ln(0.70) \Rightarrow y^2 = -\ln(0.70) \Rightarrow y = \pm\sqrt{-\ln(0.70)}$

$y = 0.5972$

note: we consider only $y = +\sqrt{-\ln(0.70)}$ as if $y < 0$, $f(y) = 0$

Question 4.12

Question: The length of time to failure (in hundreds of hours) for a transistor is a random variable Y with distribution function

$$F(y) = \begin{cases} 0, & y < 0, \\ 1 - e^{-y^2}, & y \geq 0. \end{cases}$$

(c) Find $f(y)$.

(d) Find the probability that the transistor operates for at least 200 hours.

(e) Find $P(Y > 100 \mid Y \leq 200)$.

c.) $f(y) = \frac{dF(y)}{dy} \Rightarrow f(y) = \begin{cases} 2ye^{-y^2}, & y \geq 0 \\ 0, & \text{elsewhere} \end{cases}$

$\frac{d}{dy}(1-e^{-y^2}) = 0 - e^{-y^2} \cdot -2y$
$= 2ye^{-y^2}$

d.) $P(Y \geq 200) = 1 - P(Y < 200) = 1 - F(2) = 1 - (1 - e^{-(2)^2}) = 0.0183$

e.) $P(Y > 100 \mid Y \leq 200) = \frac{P(Y > 100 \wedge Y \leq 200)}{P(Y \leq 200)} = \frac{F(2) - F(1)}{0.98168} = \frac{0.98168 - 0.63212}{0.98168} = 0.3561$

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Question 4.17

Question: The length of time required by students to complete a one-hour exam is a random variable with density

$$f(y) = \begin{cases} c\,y^2 + y, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find c.

(b) Find $F(y)$.

(c) Graph $f(y)$ and $F(y)$.

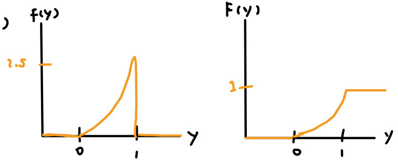
a.) $\int_{-\infty}^{\infty} cy^2+y\,dy = \int_0^1 cy^2+y\,dy = 1 \Rightarrow \frac{cy^3}{3}+\frac{y^2}{2}\Big|_0^1 = 1$

$\Rightarrow \frac{c}{3}+\frac{1}{2} = 1 \Rightarrow \frac{c}{3} = \frac{1}{2} \Rightarrow c = \frac{3}{2}$

b.) $F(y) = \int_0^y \frac{3}{2}y^2+y\,dy = \frac{y^3}{2}+\frac{y^2}{2}\Big|_0^y = \frac{y^3}{2}+\frac{y^2}{2}$

$$F(y) = \begin{cases} 0, & y<0 \\ \frac{y^3+y^2}{2}, & 0 \le y \le 1 \\ 1, & y>1 \end{cases}$$

c.)



Try graphing from Desmos

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$$F(y) = \begin{cases} 0, & y < 0 \\ \frac{y^3+y^2}{2}, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$$

(d) Using part (b), find $F(-1)$, $F(0)$, and $F(1)$.

(e) Find the probability that a randomly selected student will finish in less than half an hour.

(f) Given that a particular student needs at least 15 minutes to complete the exam, find the probability that she will require at least 30 minutes to finish.

d.) $F(-1) = 0$, $F(0) = 0$, $F(1) = 1$

e.) $P(Y < 0.5) = 0.1875$

f.) $P(Y \ge 0.5 \mid Y \ge 0.25) = 1 - P(Y < 0.5 \mid Y \ge 0.25) = 1 - \frac{P(Y < 0.5 \wedge Y \ge 0.25)}{P(Y \ge 0.25)} = 1 - \frac{F(0.5) - F(0.25)}{1 - F(0.25)}$

$= 1 - \frac{0.1875 - 0.0390625}{1 - 0.0390625} = 0.8455$

Question 4.18

Question: Let Y have the density function

$$f(y) = \begin{cases} 0.2, & -1 < y \leq 0, \\ 0.2 + c\,y, & 0 < y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find c.

(b) Find $F(y)$.

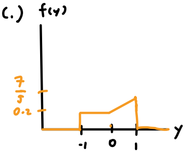
(c) Graph $f(y)$ and $F(y)$.

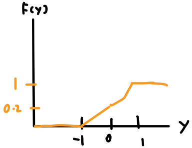
a.) $\int_{-\infty}^{\infty} f(y)\,dy = \int_{-1}^{0} 0.2\,dy + \int_{0}^{1} 0.2 + cy\,dy = 1$

$\Rightarrow \quad 0.2y\Big|_{-1}^{0} + \left[0.2y + c\frac{y^2}{2}\right]_0^1 = 1$

$\Rightarrow \quad 0.2 + 0.2 + \frac{c}{2} = 1 \Rightarrow c = \frac{6}{5}$

c.)





b.) $F(y) = \int_{0}^{y} f(y)\,dy$.

From $-1 \leq y \leq 0$: $\int_{-1}^{y} 0.2\,dy = 0.2y\Big|_{-1}^{y} = 0.2y + 0.2$

From $0 \leq y \leq 1$: $F(0) + \int_0^y 0.2 + \frac{6}{5}y\,dy = 0.2 + 0.2y + \frac{3}{5}y^2\Big|_0^y = 0.2y + \frac{3}{5}y^2 + 0.2$

$$F(y) = \begin{cases} 0, & y < -1 \\ 0.2y + 0.2, & -1 \leq y \leq 0 \\ 0.2 + 0.2y + \frac{3}{5}y^2, & 0 \leq y \leq 1 \\ 1, & y > 1 \end{cases}$$

Question 4.18

Question: Let Y have the density function

$$F(y) = \begin{cases} 0, & y < -1 \\ 0.2y + 0.2, & -1 \leq y \leq 0 \\ 0.2 + 0.2y + \frac{2}{5}y^2, & 0 \leq y \leq 1 \\ 2, & y > 1 \end{cases}$$

$$f(y) = \begin{cases} 0.2, & -1 < y \leq 0, \\ 0.2 + c\,y, & 0 < y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(d) Using part (b), find $F(-1)$, $F(0)$, and $F(1)$.

(e) Find $P(0 \leq Y \leq 0.5)$.

(f) Find $P(Y > 0.5 \mid Y > 0.1)$.

d.) $F(-1) = 0$, $F(0) = 0.2$, $F(1) = 1$

e.) $P(0 \leq Y \leq 0.5) = P(Y \leq 0.5) - P(Y \leq 0) = F(0.5) - F(0) = \frac{9}{20} - 0.2 = 0.25$

f.) $P(Y > 0.5 \mid Y > 0.1) = 1 - P(Y < 0.5 \mid Y > 0.1) = 1 - \dfrac{P(0.1 \leq Y \leq 0.5)}{1 - P(Y \leq 0.1)} = 1 - \dfrac{F(0.5) - F(0.1)}{1 - F(0.1)} = 1 - \dfrac{\frac{9}{20} - \frac{113}{500}}{1 - \frac{113}{500}}$

$= 0.7106$