ECO227Y5 Tutorial 7

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Question 4.28

Question: The proportion of time per day that all checkout counters in a supermarket are busy is a random variable Y with density function

$$f(y) = \begin{cases} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}$$

$$f(y) = \begin{cases} c\,y^2(1-y)^4, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Recall: $\Gamma(n) = (n-1)!$

(a) Find the value of c that makes $f(y)$ a probability density function.

(b) Find $\mathbb{E}[Y]$.

Recall: Beta Random variable: $B(\alpha,\beta) = \int_0^1 y^{\alpha-1}(1-y)^{\beta-1}\,dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

a.) $f(y) = \int_0^1 c\,y^2(1-y)^4\,dy = 1 \Rightarrow c\int_0^1 y^2(1-y)^4\,dy = 1 \Rightarrow c \cdot B(3,5) = 1$, $B(3,5) = \frac{\Gamma(3)\Gamma(5)}{\Gamma(8)} = \frac{2!\cdot 4!}{7!} = \frac{1}{105}$

$\Rightarrow \frac{c}{105} = 1 \Rightarrow c = 105$

Alternatively, you can expand $y^2(1-y)^4$

b.) $E(Y) = \int_{-\infty}^{\infty} y\,f(y)\,dy = 105\int_0^1 y^3(1-y)^4\,dy = 105\cdot B(4,5) = 105\cdot\frac{\Gamma(4)\Gamma(5)}{\Gamma(9)} = 105\cdot\frac{3!\,4!}{8!} = \frac{105}{280} = \frac{3}{8}$

Question 4.33

Question: Daily total solar radiation for a specified location in Florida in October has probability density function

$$f(y) = \begin{cases} \frac{3}{32}(y-2)(6-y), & 2 \leq y \leq 6, \\ 0, & \text{elsewhere,} \end{cases}$$

where measurements are in hundreds of calories.

Find the expected daily solar radiation for October.

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy = \frac{3}{32}\int_2^6 y(y-2)(6-y)\,dy = \frac{3}{32}\int_2^6 y(6y - y^2 - 12 + 2y)\,dy$$

$$= \frac{3}{32}\int_2^6 y(-y^2 + 8y - 12)\,dy = \frac{3}{32}\int_2^6 -y^3 + 8y^2 - 12y\,dy = \frac{3}{32}\left[-\frac{y^4}{4} + \frac{8y^3}{3} - 6y^2\right]\Bigg|_2^6$$

$= 4$ hundreds of calories

Question 4.42

Question: The median of the distribution of a continuous random variable Y is the value $\phi_{0.5}$ such that

$$P(Y \leq \phi_{0.5}) = 0.5.$$

What is the median of the uniform distribution on the interval (θ_1, θ_2)?

$$P(Y \leq \phi_{0.5}) = 0.5 \Rightarrow \frac{\phi_{0.5} - \theta_1}{\theta_2 - \theta_1} = 0.5$$

Recall:

$$F(y) = \begin{cases} 0 & y < \theta_1 \\ \frac{y-\theta_1}{\theta_2-\theta_1} & \theta_1 \leq y \leq \theta_2 \\ 1 & y > \theta_2 \end{cases}$$

$$\Rightarrow \phi_{0.5} = 0.5(\theta_2 - \theta_1) + \theta_1$$

$$\phi_{0.5} = \frac{\theta_2 - \theta_1}{2} + \theta_1$$

$$\phi_{0.5} = \frac{\theta_2 - \theta_1 + 2\theta_1}{2} = \frac{\theta_1 + \theta_2}{2}$$

Question 4.45

Question: Upon studying low bids for shipping contracts, a microcomputer manufacturing company finds that intrastate contracts have low bids that are uniformly distributed between 20 and 25 (in units of thousands of dollars). Find the probability that the low bid on the next intrastate shipping contract:

(a) is below \$22,000.

(b) is in excess of \$24,000.

$$F(y) = \begin{cases} 0, & y < 20 \\ \frac{y-20}{5} & 20 \le y \le 25 \\ 1, & y > 25 \end{cases}$$

$Y \sim \text{Unif}(20, 25)$

a.) $P(Y < 22) = P(Y \le 22) = F(22) = \frac{22-20}{5} = \frac{2}{5}$

b.) $P(Y > 24) = 1 - P(Y \le 24) = 1 - F(24) = 1 - \frac{24-20}{5} = \frac{1}{5}$

Question 4.60

Question: A normally distributed random variable has density function

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}, \quad -\infty < y < \infty.$$

Using the fundamental properties associated with any density function, argue that the parameter σ must be such that $\sigma > 0$.

$f(y) \geq 0$ at all y's

Notice that $2\pi > 0$, so $\sqrt{2\pi} > 0$, e^{λ} for any $\lambda \in \mathbb{R} > 0$ so we need not worry about

$e^{\frac{-(y-\mu)^2}{2\sigma^2}}$ being negative so we need $\frac{1}{\sigma\sqrt{2\pi}} > 0$, clearly $\sigma > 0$ fulfills this condition.

If $\sigma = 0$, denominator $= 0$

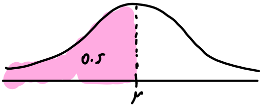
Question: What is the median of a normally distributed random variable with mean μ and standard deviation σ?

$Y \sim N(\mu, \sigma^2)$

$P(Y \leq \mu) = 0.5 = P(Y \geq \mu)$

by symmetry of normal,

so $\phi_{0.5} = \mu$

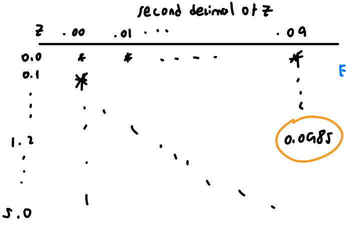


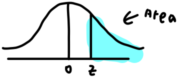
Question: If Z is a standard normal random variable, find:

(a) $P(Z^2 < 1)$

(b) $P(Z^2 < 3.84146)$

Reading Z Table:





Ex. $P(Z \le 1.29)$

$= 1 - P(Z > 1.29)$

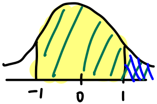
find this on your table

$= 1 - 0.0985$

a.) $P(Z^2 < 1)$

$= P(Z < \pm\sqrt{1})$

$= P(-1 < Z < 1)$



not on table

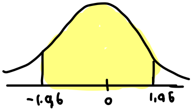
$= P(Z > -1) - P(Z > 1)$

$= P(Z < 1) - 0.1587$

$= 1 - P(Z > 1) - 0.1587$

$= 1 - 0.1587 - 0.01587 = 0.6826$

b.) $P(Z^2 < 3.84146)$



$= P(Z < \pm\sqrt{3.84146}) = P(-1.96 < Z < 1.96)$

$= P(Z > -1.96) - P(Z > 1.96) = P(Z < 1.96) - 0.0250 = 1 - P(Z > 1.96) - 0.0250$

$= 1 - 0.0250 - 0.0250 = 0.95$

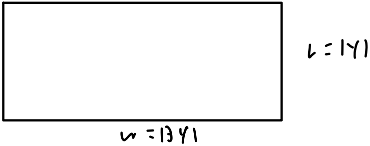
Question 4.80

Question: Assume that Y is normally distributed with mean μ and standard deviation σ. After observing a value of Y, a mathematician constructs a rectangle with length $L = |Y|$ and width $W = 3|Y|$. Let A denote the area of the resulting rectangle.

What is $\mathbb{E}[A]$?

Note: $|ab| = |a| \cdot |b|$ for all $a, b \in \mathbb{R}$

$Y \sim N(\mu, \sigma^2)$



$A = L \cdot w = |Y| \cdot |3Y| = |3Y^2|$

$A = 3|Y^2| = 3Y^2$ as $Y^2 \geq 0$ already

$E(A) = E(3Y^2) = 3E(Y^2) = 3(\sigma^2 + \mu^2)$

Recall: $V(Y) = \sigma^2 = E(Y^2) - E(Y)^2$

$\Rightarrow \sigma^2 = E(Y^2) - \mu^2$

$\Rightarrow E(Y^2) = \sigma^2 + \mu^2$