ECO227Y5 Tutorial 8

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Question 4.96

Question: Suppose that a random variable Y has a probability density function

Recall: $Y \sim \chi^2_\nu$ if $\beta = 2, \alpha = \frac{\nu}{2}$

$$f(y) = \begin{cases} k\,y^3 e^{-y/2}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of k that makes $f(y)$ a probability density function.

(b) Does Y have a χ^2 distribution? If so, how many degrees of freedom?

(c) What are the mean and standard deviation of Y?

a.) $e^{-y/2}$ gives us that $\beta = 2$ and y^3 gives us that $\alpha = 3+1 = 4$

so we need $k = \frac{1}{2^4 \cdot \Gamma(4)} = \frac{1}{16 \cdot 3!} = \frac{1}{96}$ for $f(y)$ to integrate to 1.

Recall: For $Y \sim \text{Gamma}(\alpha, \beta)$

$E(Y) = \alpha\beta$

$V(Y) = \alpha\beta^2$

b.) Yes! $\beta = 2$, $\alpha = \frac{\nu}{2} \Rightarrow 4 = \frac{\nu}{2} \Rightarrow \nu = 8$ degrees of freedom

c.) $E(Y) = \alpha \cdot \beta = 4 \cdot 2 = 8$

$V(Y) = 4 \cdot 4 = 16 \Rightarrow SD(Y) = 4$

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Question 4.111

Question: Suppose that Y has a gamma distribution with parameters α and β.

(a) If a is any positive or negative value such that $\alpha + a > 0$, show that

Recall: $Y \sim \text{Gamma}(\alpha, \beta)$

$$f(y) = \begin{cases} \dfrac{y^{\alpha-1} e^{-y/\beta}}{\beta^{\alpha}\,\Gamma(\alpha)}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

$$\mathbb{E}(Y^a) = \beta^a \frac{\Gamma(\alpha + a)}{\Gamma(\alpha)}.$$

(b) Why did your answer in part (a) require that $\alpha + a > 0$?

(c) Show that, with $a = 1$, the result in part (a) gives $\mathbb{E}(Y) = \alpha\beta$.

a.) Proof: let $\alpha + a > 0$, $Y \sim \text{Gamma}(\alpha, \beta)$

$$E(Y^a) = \int_0^\infty \frac{y^{a+\alpha-1} e^{-y/\beta}}{\beta^{\alpha}\,\Gamma(\alpha)}\,dy = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \underbrace{\int_0^\infty y^{a+\alpha-1} e^{-y/\beta}\,dy}_{=\ \beta^{a+\alpha}\cdot\Gamma(a+\alpha)}$$

This looks like the numerator of $f(y)$ for $Y \sim \text{Gamma}(\alpha+a, \beta)$

$$E(Y^a) = \frac{\beta^{a+\alpha}\,\Gamma(a+\alpha)}{\beta^{\alpha}\,\Gamma(\alpha)} = \beta^a \frac{\Gamma(\alpha+a)}{\Gamma(\alpha)} \quad \blacksquare$$

b.) To be a valid gamma distribution, we need α to be > 0, if $Y \sim \text{Gamma}(\alpha+a, \beta)$ then $\alpha + a > 0$ is needed.

c.) $$E[Y^a] = \beta^a \frac{\Gamma(\alpha+a)}{\Gamma(\alpha)} \Rightarrow E[Y^1] = \beta \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = \beta \frac{\alpha!}{(\alpha-1)!} = \beta \frac{\alpha \cdot \cancel{\alpha-1} \cdots}{\cancel{\alpha-1} \cdots} = \beta\alpha.$$

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Question 4.111

Question: Suppose that Y has a gamma distribution with parameters α and β.

(d) Use the result in part (a) to give an expression for $\mathbb{E}(\sqrt{Y})$. What do you need to assume about α?

(e) Use the result in part (a) to give an expression for $\mathbb{E}(1/Y)$, $\mathbb{E}(1/\sqrt{Y})$, and $\mathbb{E}(1/Y^2)$. What do you need to assume about α in each case?

d.) $E[Y^{1/2}] = \beta^{1/2}\dfrac{\Gamma(1/2+\alpha)}{\Gamma(\alpha)}$, need $\alpha + \frac{1}{2} > 0 \Rightarrow \alpha > -\frac{1}{2}$, but $\alpha > 0$ anyways so it is already satisfied

e.) $E(\overset{\sim Y^{-1}}{\tfrac{1}{Y}}) = \beta^{-1}\dfrac{\Gamma(\alpha-1)}{\Gamma(\alpha)}$, need $\alpha - 1 > 0 \Rightarrow \alpha > 1$

$E[\overset{\sim Y^{-1/2}}{\tfrac{1}{\sqrt{Y}}}] = \beta^{-1/2}\dfrac{\Gamma(\alpha - 1/2)}{\Gamma(\alpha)}$, need $\alpha - 1/2 > 0 \Rightarrow \alpha > \frac{1}{2}$

$E(\overset{\sim Y^{-2}}{\tfrac{1}{Y^2}}) = \beta^{-2}\dfrac{\Gamma(\alpha-2)}{\Gamma(\alpha)}$, need $\alpha - 2 > 0 \Rightarrow \alpha > 2$

Question 4.112

Question: Suppose that Y has a χ^2 distribution with ν degrees of freedom. Use the results in Exercise 4.111 in your answers to the following. These results will be useful when we study the t and F distributions in Chapter 7.

(a) Give an expression for $\mathbb{E}(Y^a)$ if $\nu > -2a$.

(b) Why did your answer in part (a) require that $\nu > -2a$?

(c) Use the result in part (a) to give an expression for $\mathbb{E}(\sqrt{Y})$. What do you need to assume about ν?

(d) Use the result in part (a) to give an expression for $\mathbb{E}(1/Y)$, $\mathbb{E}(1/\sqrt{Y})$, and $\mathbb{E}(1/Y^2)$. What do you need to assume about ν in each case?

a.) $\chi^2 \Rightarrow \beta = 2$, $\alpha = \frac{v}{2}$

$$E(Y^a) = 2^a \frac{\Gamma(\frac{v}{2}+a)}{\Gamma(\frac{v}{2})}$$

b.) As we need $\frac{v}{2}+a > 0 \Rightarrow v > -2a$

c.) $$E(\sqrt{Y}) = E(Y^{1/2}) = 2^{1/2}\frac{\Gamma(\frac{v}{2}+\frac{1}{2})}{\Gamma(\frac{v}{2})} = 2^{1/2}\frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})}$$

$\frac{v+1}{2} > 0 \Rightarrow v > -1$ since $v > 0$ for χ^2, this is always satisfied

d.) $$E\left(\frac{1}{Y}\right) = 2^{-1}\frac{\Gamma(\frac{v}{2}-1)}{\Gamma(\frac{v}{2})}, \quad v > 2$$

$$E\left(\frac{1}{\sqrt{Y}}\right) = 2^{-1/2}\frac{\Gamma(\frac{v}{2}-\frac{1}{2})}{\Gamma(\frac{v}{2})}, \quad v > 1$$

$$E\left(\frac{1}{Y^2}\right) = 2^{-2}\frac{\Gamma(\frac{v}{2}-2)}{\Gamma(\frac{v}{2})}, \quad v > 4$$

Question 4.128

Question: Suppose that a random variable Y has a probability density function

$$f(y) = \begin{cases} 6y(1-y), & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find $F(y)$.

(b) Graph $F(y)$ and $f(y)$.

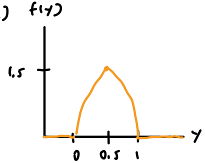
(c) Find $P(0.5 \leq Y \leq 0.8)$.

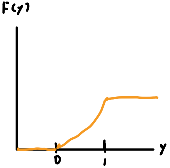
a.) $F(y) = \int_0^y 6y(1-y)\,dy = \int_0^y 6y - 6y^2\,dy = 3y^2 - 2y^3 \Big|_0^y$

$= 3y^2 - 2y^3$

$$F(y) = \begin{cases} 0, & y < 0 \\ 3y^2 - 2y^3, & 0 \leq y \leq 1 \\ 1, & y > 1 \end{cases}$$

b.)





c.) $P(0.5 \leq Y \leq 0.8) = P(Y \leq 0.8) - P(Y \leq 0.5)$

$= F(0.8) - F(0.5)$

$= \frac{112}{125} - \frac{1}{2}$

$= 0.396$

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Question 4.131

Question: Errors in measuring the time of arrival of a wave front from an acoustic source sometimes have an approximate beta distribution. Suppose that these errors, measured in microseconds, have approximately a beta distribution with $\alpha = 1$ and $\beta = 2$.

(a) What is the probability that the measurement error in a randomly selected instance is less than 0.5 μs?
microseconds

(b) Give the mean and standard deviation of the measurement errors.

Recall:

$$f(y) = \begin{cases} \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha,\beta)}, & 0 \le y \le 1 \\ 0, & \text{elsewhere} \end{cases}, \quad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

a.) $Y \sim \text{Beta}(1,2)$

$$B(1,2) = \frac{\Gamma(1)\Gamma(2)}{\Gamma(3)} = \frac{1 \cdot 1}{2} = \frac{1}{2}$$

So $f(y) = 2(1-y)$

$$P(Y < 0.5) = \int_0^{0.5} 2 - 2y \, dy = 2y - y^2 \Big|_0^{0.5} = 1 - 0.25 = 0.75$$

Recall:

$$E(Y) = \frac{\alpha}{\alpha+\beta} \qquad V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

b.) $E(Y) = \frac{1}{3}$

$$V(Y) = \frac{2}{3^2 \cdot 4} = \frac{1}{18} \Rightarrow SD(Y) = \sqrt{\frac{1}{18}}$$

Question 4.133

Question: The proportion of time per day that all checkout counters in a supermarket are busy is a random variable Y with density

$$f(y) = \begin{cases} c\,y^2(1-y)^4, & 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of c that makes $f(y)$ a probability density function.

(b) Find $\mathbb{E}[Y]$. (Use what you have learned about the beta-type distribution. Compare your answers to those obtained in Exercise 4.28.)

(c) Calculate the standard deviation of Y.

a.) $f(y) = c\int_0^1 y^2(1-y)^4\,dy \rightarrow \beta(3,5)$ so $c = \frac{1}{\beta(3,5)} = \frac{1}{\frac{\Gamma(3)\Gamma(5)}{\Gamma(8)}} = \frac{1}{\frac{1}{105}} \Rightarrow c = 105$

b.) $E(Y) = \frac{\alpha}{\alpha+\beta} = \frac{3}{8}$

c) $V(Y) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{15}{8^2 \cdot 9} = \frac{5}{192} \Rightarrow SD(Y) = \sqrt{\frac{5}{192}} = 0.161$