ECO227Y5 Tutorial 9

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Question 5.3

Question: Of nine executives in a business firm, four are married, three have never married, and two are divorced. Three of the executives are to be selected for promotion.

Let Y_1 denote the number of married executives and Y_2 denote the number of never-married executives among the three selected for promotion. Assuming that the three are randomly selected from the nine available, find the joint probability function of Y_1 and Y_2.

Note: Y_1 and Y_2 can't be both 0!

$Y_1 \in \{0, 1, 2, 3\}$ # Divorce $= 3 - Y_1 - Y_2$

$Y_2 \in \{0, 1, 2, 3\}$ $1 \le Y_1 + Y_2 \le 3$

Total ways to choose 3 out of 9 people: $\binom{9}{3} = 84$

$$p(y_1, y_2) = \frac{\underbrace{\binom{4}{y_1}}_{\text{married}} \underbrace{\binom{3}{y_2}}_{\text{never married}} \underbrace{\binom{2}{3-y_1-y_2}}_{\text{Divorced}}}{\binom{9}{3}}$$

Question 5.5

Question: Refer to Example 5.4. The joint density of Y_1, the proportion of the capacity of the tank that is stocked at the beginning of the week, and Y_2, the proportion of the capacity sold during the week, is given by

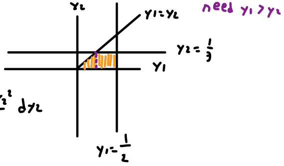
$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \leq y_2 \leq y_1 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find $F\left(\frac{1}{2}, \frac{1}{3}\right) = P\left(Y_1 \leq \frac{1}{2},\ Y_2 \leq \frac{1}{3}\right)$.

a.) $F\left(\frac{1}{2}, \frac{1}{3}\right) = P\left(Y_1 \leq \frac{1}{2}, Y_2 \leq \frac{1}{3}\right) \ni \begin{array}{l} y_2 \leq y_1 \leq \frac{1}{2} \\ 0 \leq y_2 \leq \frac{1}{3} \end{array}$

$$= \int_{y_2=0}^{\frac{1}{3}} \int_{y_1=y_2}^{\frac{1}{2}} 3y_1 \, dy_1 dy_2 = \int_0^{\frac{1}{3}} \frac{3y_1^2}{2} \Bigg|_{y_2}^{\frac{1}{2}} dy_2 = \int_0^{\frac{1}{3}} \frac{3}{8} - \frac{3y_2^2}{2} \, dy_2$$

$$= \frac{3}{8} y_2 - \frac{y_2^3}{2} \Bigg|_0^{\frac{1}{3}} = \frac{1}{8} - \frac{1}{54} = \frac{23}{216}$$



Alternatively:

$$= \int_{y_1=0}^{\frac{1}{3}} \int_{y_2=0}^{y_2=y_1} 3y_1 \, dy_2 dy_1 + \int_{y_1=\frac{1}{3}}^{\frac{1}{2}} \int_{y_2=0}^{y_2=\frac{1}{3}} 3y_1 \, dy_2 dy_1$$

Question 5.5

Question: Refer to Example 5.4. The joint density of Y_1, the proportion of the capacity of the tank that is stocked at the beginning of the week, and Y_2, the proportion of the capacity sold during the week, is given by

$$f(y_1, y_2) = \begin{cases} 3y_1, & 0 \le y_2 \le y_1 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

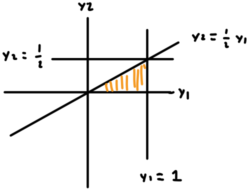
(b) Find $P(Y_2 \le Y_1/2)$, the probability that the amount sold is less than half the amount purchased.

$y = \frac{x}{2}$

$$P(Y_2 \le Y_1/2) = \int_{y_1=0}^{1} \int_{y_2=0}^{y_2=\frac{1}{2}y_1} 3y_1 \, dy_2 \, dy_1$$

$$= \int_0^1 3y_1 y_2 \Big|_0^{\frac{1}{2}y_1} dy_1 = \int_0^1 \frac{3}{2} y_1^2 \, dy_1$$

$$= \frac{y_1^3}{2} \Big|_0^1 = \frac{1}{2}$$



Alternatively:

$$= \int_{y_2=0}^{\frac{1}{2}} \int_{y_1=2y_2}^{y_1=1} 3y_1 \, dy_1 \, dy_2$$

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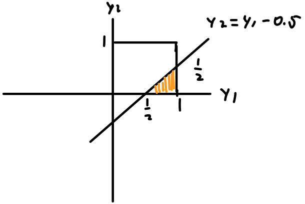
Question 5.6

Question: Refer to Example 5.3. If a radioactive particle is randomly located in a square of unit length, a reasonable model for the joint density function for Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} 1, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What is $P(Y_1 - Y_2 > 0.5)$?



$$P(Y_1 - Y_2 > 0.5) = P(Y_1 - 0.5 > Y_2)$$

$$= \int_{y_1=0.5}^{1} \int_{y_2=0}^{y_2=y_1-0.5} 1\, dy_2\, dy_1 = \int_{0.5}^{1} y_2 \Big|_0^{y_1-0.5} dy_1$$

$$= \int_{0.5}^{1} y_1 - 0.5\, dy_1 = \frac{y_1^2}{2} - 0.5 y_1 \Big|_{0.5}^{1} = \frac{1}{8}$$

Alternatively:

$$\int_{y_2=0}^{0.5} \int_{y_1=y_2+0.5}^{y_1=1} 1\, dy_1\, dy_2$$

Question 5.6

Question: Refer to Example 5.3. If a radioactive particle is randomly located in a square of unit length, a reasonable model for the joint density function for Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} 1, & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

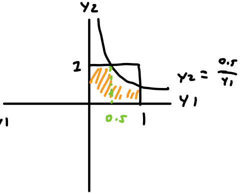
(b) What is $P(Y_1 Y_2 < 0.5)$?

$$P(Y_1Y_2 < 0.5) = P\left(Y_2 \le \frac{0.5}{Y_1}\right)$$

$$= \int_{y_1=0.5}^{1} \int_{y_2=0}^{y_2=\frac{0.5}{y_1}} 1\, dy_2\, dy_1 + \int_{y_1=0}^{0.5} \int_{y_2=0}^{y_2=1} 1\, dy_2\, dy_1$$

$$= \int_0^1 \frac{0.5}{y_1}\, dy_1 + \int_0^{0.5} 1\, dy_1$$

$$= 0.5 \ln y_1 \Big|_{0.5}^{1} + 0.5 = -0.5 \ln 0.5 + 0.5$$



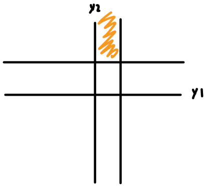
Alternatively: $\displaystyle\int_0^{0.5} \int_{y_1=0}^{y_1=1} 1\, dy_1\, dy_2 + \int_{0.5}^{1} \int_0^{y_1=\frac{0.5}{y_2}} 1\, dy_1\, dy_2$

Question 5.7

Question: Let Y_1 and Y_2 have joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) What is $P(Y_1 < 1,\ Y_2 > 5)$?



$$P(Y_1 < 1, Y_2 > 5) = \int_{y_2=5}^{y_2=\infty}\int_{y_1=0}^{y_1=1} e^{-(y_1+y_2)}\, dy_1\, dy_2$$

$$= \int_{y_2=5}^{\infty}\int_{y_1=0}^{1} e^{-y_1}\cdot e^{-y_2}\, dy_1\, dy_2 = \int_{y_2=5}^{\infty} e^{-y_2}\int_{y_1=0}^{1} e^{-y_1}\, dy_1\, dy_2$$

$$= \int_{y_2=5}^{\infty} e^{-y_2}\left[-e^{-y_1}\right]\Big|_0^1 dy_2 = \int_5^{\infty} e^{-y_2}\left(-e^{-1}+1\right) dy_2$$

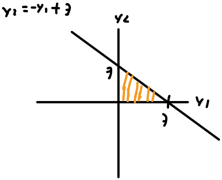
$$= (1-e^{-1})\int_5^{\infty} e^{-y_2}\, dy_2 = (1-e^{-1})\left[-e^{-y_2}\right]_5^{\infty} = (1-e^{-1})e^{-5}$$

Question 5.7

Question: Let Y_1 and Y_2 have joint density function

$$f(y_1, y_2) = \begin{cases} e^{-(y_1+y_2)}, & y_1 > 0,\ y_2 > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(b) What is $P(Y_1 + Y_2 < 3)$?



$$P(Y_1 + Y_2 < 3) = P(Y_2 < -Y_1 + 3)$$

$$= \int_{y_1=0}^{3} \int_{y_2=0}^{y_2=-y_1+3} e^{-(y_1+y_2)}\, dy_2\, dy_1$$

$$= \int_0^3 e^{-y_1} \left[-e^{-y_2}\right]_0^{-y_1+3} dy_1 = \int_0^3 e^{-y_1}\left(-e^{y_1-3}+1\right) dy_1 = \int_0^3 -e^{-3} + e^{-y_1}\, dy_1$$

$$= \left[-e^{-3} y_1 - e^{-y_1}\right]_0^3 = -3e^{-3} - e^{-3} + 1 = 1 - 4e^{-3}$$

Alternatively: $\int_0^3 \int_{y_1=0}^{y_1=3-y_2} e^{-(y_1+y_2)}\, dy_1\, dy_2$

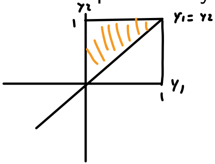
Question 5.9

Question: Let Y_1 and Y_2 have the joint probability density function

$$f(y_1, y_2) = \begin{cases} k(1-y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the value of k that makes this a probability density function.

$$1 = k\int_0^1 \int_{y_2=y_1}^{y_2=1} 1 - y_2 \, dy_2 \, dy_1$$



$$1 = k \int_0^1 y_2 - \frac{y_2^2}{2} \Big|_{y_1}^{1} dy_1$$

Alternatively: $1 = \int_0^1 \int_{y_1=0}^{y_1=y_2} k(1-y_2)\, dy_1 \, dy_2$

$$1 = k \int_0^1 \frac{1}{2} - y_1 + \frac{y_1^2}{2} \, dy_1$$

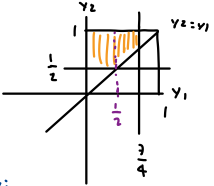
$$1 = k\left(\frac{1}{2}y_1 - \frac{y_1^2}{2} + \frac{y_1^3}{6} \right]_0^1 \Rightarrow 1 = k\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right] \Rightarrow 1 = \frac{1}{6}k \Rightarrow k=6$$

Question 5.9

Question: Let Y_1 and Y_2 have the joint probability density function

$$f(y_1, y_2) = \begin{cases} \overset{6}{k}(1 - y_2), & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(b) Find $P\left(Y_1 \le \frac{3}{4},\ Y_2 \ge \frac{1}{2}\right)$.



$$P\left(Y_1 \le \tfrac{3}{4},\ Y_2 \ge \tfrac{1}{2}\right) = \int_0^{\frac{1}{2}} \int_{y_2=\frac{1}{2}}^{y_2=1} 6(1-y_2)\,dy_2\,dy_1 + \int_{\frac{1}{2}}^{\frac{3}{4}} \int_{y_2=y_1}^{y_2=1} 6(1-y_2)\,dy_2\,dy_1$$

$$= 6\int_0^{\frac{1}{2}} y_2 - \frac{y_2^2}{2}\Big|_{\frac{1}{2}}^{1} dy_1 + 6\int_{\frac{1}{2}}^{\frac{3}{4}} y_2 - \frac{y_2^2}{2}\Big|_{y_1}^{1} dy_1$$

$$= 6\int_0^{\frac{1}{2}} \frac{1}{2} - \left(\frac{1}{2} - \frac{1}{8}\right) dy_1 + 6\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{1}{2} - y_1 + \frac{y_1^2}{2}\,dy_1$$

$$= 6\int_0^{\frac{1}{2}} \frac{1}{8}\,dy_1 + 6\left[\frac{1}{2}y_1 - \frac{y_1^2}{2} + \frac{y_1^3}{6}\right]_{\frac{1}{2}}^{\frac{3}{4}} = \frac{3}{8} + 6\left[\frac{7}{384}\right] = \frac{31}{64}$$

Alternatively:

$$P = \int_{\frac{1}{2}}^{\frac{3}{4}} \int_{y_1=0}^{y_2} 6(1-y_2)\,dy_1\,dy_2 + \int_{3/4}^{1} \int_{y_1=0}^{3/4} 6(1-y_2)\,dy_1\,dy_2$$

Question 5.15

Question: The management at a fast-food outlet is interested in the joint behavior of the random variables Y_1, defined as the total time between a customer's arrival at the store and departure from the service window, and Y_2, the time a customer waits in line before reaching the service window. Because Y_1 includes the time a customer waits in line, we must have $Y_1 \geq Y_2$.
The relative frequency distribution of observed values of Y_1 and Y_2 can be modeled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

with time measured in minutes. Find

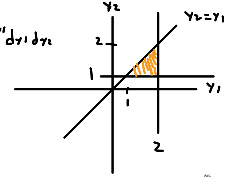
(a) $P(Y_1 < 2,\ Y_2 > 1)$.

Alternatively:

$$p = \int_1^2 \int_{y_1 = y_2}^{2} e^{-y_1} \, dy_1 \, dy_2$$

$u = y_1 - 1 \quad dv = e^{-y_1}$

$du = 1 dv \quad v = -e^{-y_1}$



$$p = \int_{y_1=1}^{2} \int_{y_2=1}^{y_2=y_1} e^{-y_1} \, dy_2 \, dy_1 = \int_1^2 e^{-y_1}(y_1 - 1) \, dy_1$$

$$p = -e^{-y_1}(y_1 - 1)\Big|_1^2 - \int_1^2 -e^{-y_1} \, dy_1 = -e^{-2} - e^{-y_1}\Big|_1^2 = -e^{-2} - e^{-2} + e^{-1} = e^{-1} - 2e^{-2}$$

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Question 5.15

Question: The relative frequency distribution of observed values of Y_1 and Y_2 can be modeled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty, \\ 0, & \text{elsewhere}, \end{cases}$$

with time measured in minutes. Find

(b) $P(Y_1 \geq 2Y_2)$.

$$P(Y_1 \geq 2Y_2) = P\left(\frac{Y_1}{2} \geq Y_2\right) = \int_0^\infty \int_{y_2=0}^{y_2=\frac{y_1}{2}} e^{-y_1}\, dy_2\, dy_1$$

$u = y_1 \quad dv = e^{-y_1}$

$du = dy_1 \quad v = -e^{-y_1}$

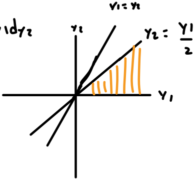
$$= \int_0^\infty e^{-y_1} \cdot \frac{y_1}{2}\, dy_1 = \frac{1}{2} \int_0^\infty e^{-y_1} \cdot y_1\, dy_1$$

$$= \frac{1}{2}\left[-y_1 e^{-y_1} \Big|_0^\infty + \int_0^\infty e^{-y_1}\, dy_1 \right] = \frac{1}{2}\left[0 - e^{-y_1} \Big|_0^\infty \right]$$

$$= \frac{1}{2}\left[1 \right] = \frac{1}{2}$$

Alternatively:

$$p = \int_0^\infty \int_{y_1=2y_2}^\infty e^{-y_1}\, dy_1\, dy_2$$



Question 5.15

Question: The relative frequency distribution of observed values of Y_1 and Y_2 can be modeled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \le y_2 \le y_1 < \infty, \\ 0, & \text{elsewhere,} \end{cases}$$

with time measured in minutes. Find

(c) $P(Y_1 - Y_2 \ge 1)$ (notice that $Y_1 - Y_2$ denotes the time spent at the service window).

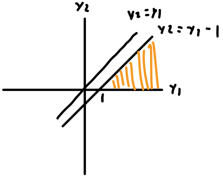
$$P(Y_1 - Y_2 \ge 1) = P(Y_2 \le Y_1 - 1)$$

$u = y_1 - 1$ $\quad dv = e^{-y_1}$

$du = dy_1$ $\quad v = -e^{-y_1}$

$$= \int_{y_1=1}^{\infty} \int_{y_2=0}^{y_2 = y_1 - 1} e^{-y_1}\, dy_2\, dy_1 = \int_1^{\infty} e^{-y_1}(y_1 - 1)\, dy_1$$

$$= -e^{-y_1}(y_1 - 1)\Big|_1^{\infty} + \int_1^{\infty} e^{-y_1}\, dy_1 = -e^{-y_1}\Big|_1^{\infty} = e^{-1}$$



Alternatively:

$$P = \int_0^{\infty} \int_{y_1 = y_2 + 1}^{\infty} e^{-y_1}\, dy_1\, dy_2$$

Extra Homework

Question: Given the joint density of (Y_1, Y_2) as

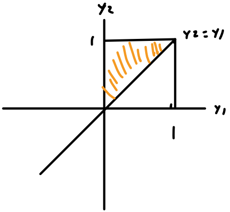
$$f(y_1, y_2) = \begin{cases} c\, y_1, & 0 \leq y_1 \leq y_2 \leq 1, \\ 0, & \text{elsewhere,} \end{cases}$$

(1) Find c.

$$1 = \int_0^1 \int_{y_2 = y_1}^{y_2 = 1} c y_1 \, dy_2 \, dy_1$$

$$1 = \int_0^1 c y_1 - c y_1^2 \, dy_1 \Rightarrow 1 = \frac{c y_1^2}{2} - \frac{c y_1^3}{3} \Big|_0^1$$

$$\Rightarrow \quad 1 = \frac{c}{2} - \frac{c}{3} \Rightarrow 1 = \frac{c}{6} \Rightarrow c = 6$$



Alternatively:

$$1 = \int_0^1 \int_{y_1 = 0}^{y_2} c y_1 \, dy_1 \, dy_2$$

Extra Homework

Question: Given the joint density of (Y_1, Y_2) as

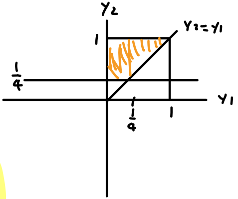
$$f(y_1, y_2) = \begin{cases} \overset{6}{c}\, y_1, & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere,} \end{cases}$$

(2) Calculate $P(0 \le Y_1 \le 1,\ Y_2 \ge \frac{1}{4})$.

$$P = \int_{\frac{1}{4}}^{1} \int_{y_1=0}^{y_2} 6y_1\, dy_1\, dy_2 = \int_0^1 3y_1^2 \Big|_0^{y_2} dy_2$$

$$= \int_{\frac{1}{4}}^{1} 3y_2^2\, dy_2$$

$$= y_2^3 \Big|_{\frac{1}{4}}^{1} = \frac{63}{64}$$



Alternatively: $P = \int_0^{\frac{1}{4}} \int_{y_2=\frac{1}{4}}^{y_2=1} 6y_1\, dy_2\, dy_1 + \int_{\frac{1}{4}}^{1} \int_{y_2=y_1}^{y_2=1} 6y_1\, dy_2\, dy_1$

Extra Homework

Question: Given the joint density of (Y_1, Y_2) as

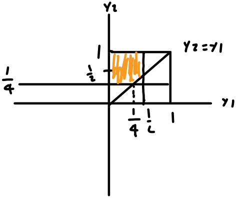
$$f(y_1, y_2) = \begin{cases} c\,y_1, & 0 \le y_1 \le y_2 \le 1, \\ 0, & \text{elsewhere}, \end{cases}$$

(3) Calculate $P(0 \le Y_1 \le 0.5,\ Y_2 \ge \frac{1}{4})$.

$$P = \int_0^{\frac{1}{4}} \int_{y_2=\frac{1}{4}}^{y_2=1} 6y_1\,dy_2\,dy_1 + \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{y_2=y_1}^{y_2=1} 6y_1\,dy_2\,dy_1$$

$$P = \int_0^{\frac{1}{4}} \frac{9}{2} y_1\,dy_1 + \int_{\frac{1}{4}}^{\frac{1}{2}} 6y_1 - 6y_1^2\,dy_1$$

$$P = \frac{9y_1^2}{4}\Bigg|_0^{\frac{1}{4}} + \left(3y_1^2 - 2y_1^3\right)\Bigg|_{\frac{1}{4}}^{\frac{1}{2}} = \frac{9}{64} + \frac{11}{32} = \frac{31}{64}$$



Alternatively:

$$P = \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{y_1=0}^{y_2} 6y_1\,dy_1\,dy_2 + \int_{\frac{1}{2}}^{1} \int_{y_1=0}^{1/2} 6y_1\,dy_1\,dy_2$$